

Chapter 3

Connections between dynamical systems and crossed products of Banach algebras by \mathbb{Z}

*This chapter has been published as: Svensson, C., Silvestrov S., de Jeu M., “Connections between dynamical systems and crossed products of Banach algebras by \mathbb{Z} ”, in “Methods of Spectral Analysis in Mathematical Physics”, Janas, J., Kurasov, P., Laptev, A., Naboko, S., Stolz, G. (Eds.), Operator Theory: Advances and Applications **186**, Birkhäuser, Basel, 2009, 391-401.*

Abstract. Starting with a complex commutative semi-simple completely regular Banach algebra A and an automorphism σ of A , we form the crossed product of A by the integers, where the latter act on A via iterations of σ . The automorphism induces a topological dynamical system on the character space $\Delta(A)$ of A in a natural way. We prove equivalence between the property that every non-zero ideal of the crossed product has non-zero intersection with the subalgebra A , maximal commutativity of A in the crossed product, and density of the aperiodic points of the induced system on the character space. We also prove that every non-trivial ideal of the crossed product always intersects the commutant of A non-trivially. Furthermore, under the assumption that A is unital and such that $\Delta(A)$ consists of infinitely many points, we show equivalence between simplicity of the crossed product and minimality of the induced system, and between primeness of the crossed product and topological transitivity of the system.

3.1. Introduction

A lot of work has been done in the direction of connections between certain topological dynamical systems and crossed product C^* -algebras. In [6] and [7], for example, one starts with a homeomorphism σ of a compact Hausdorff space X and constructs the crossed prod-

uct C^* -algebra $C(X) \rtimes_{\alpha} \mathbb{Z}$, where $C(X)$ is the algebra of continuous complex-valued functions on X and α is the automorphism of $C(X)$ naturally induced by σ . One of many results obtained is equivalence between simplicity of the algebra and minimality of the system, provided that X consists of infinitely many points, see [1], [3], [6], [7] or, for a more general result, [8]. In [5], a purely algebraic variant of the crossed product is considered, with more general classes of algebras than merely continuous functions on compact Hausdorff spaces serving as “coefficient algebras”. For example, it was proved there that, for such crossed products, the analogue of the equivalence between density of aperiodic points of a dynamical system and maximal commutativity of the “coefficient algebra” in the associated crossed product C^* -algebra is true for significantly larger classes of coefficient algebras and associated dynamical systems. In this paper, we go beyond these results and investigate the ideal structure of some of the crossed products considered in [5]. More specifically, we consider crossed products of complex commutative semi-simple completely regular Banach algebras A by the integers under an automorphism $\sigma : A \rightarrow A$.

In Section 3.2 we give the most general definition of the kind of crossed product that we will use throughout this paper. We also mention the elementary result that the commutant of the coefficient algebra is automatically a commutative subalgebra of the crossed product. The more specific setup that we will be working in is introduced in Section 3.3. There we also introduce some notation and mention two basic results concerning a canonical isomorphism between certain crossed products, and an explicit description of the commutant of the coefficient algebra in one of them.

According to [7, Theorem 5.4], the following three properties are equivalent:

- The aperiodic points of (X, σ) are dense in X ;
- Every non-zero closed ideal I of the crossed product C^* -algebra $C(X) \rtimes_{\alpha} \mathbb{Z}$ is such that $I \cap C(X) \neq \{0\}$;
- $C(X)$ is a maximal abelian C^* -subalgebra of $C(X) \rtimes_{\alpha} \mathbb{Z}$.

In Section 3.4, an analogue of this result is proved for our setup. A reader familiar with the theory of crossed product C^* -algebras will easily recognize that if one chooses $A = C(X)$ for X a compact Hausdorff space in Corollary 3.4.5 below, then the crossed product is canonically isomorphic to a norm-dense subalgebra of the crossed product C^* -algebra coming from the considered induced dynamical system. We also combine this with a theorem from [5] to conclude a stronger result for the Banach algebra $L_1(G)$, where G is a locally compact abelian group with connected dual group.

In Section 3.5, we prove the equivalence between algebraic simplicity of the crossed product and minimality of the induced dynamical system in the case when A is unital with its character space consisting of infinitely many points. This is analogous to [7, Theorem 5.3], [1, Theorem VIII 3.9], the main result in [3] and, as a special case of a more general result, [8, Corollary 8.22] for the crossed product C^* -algebra.

In Section 3.6, the fact that the commutant of A always has non-zero intersection with every non-zero ideal of the crossed product is shown. This should be compared with the fact that A itself may well have zero intersection with such ideals, as Corollary 3.4.5 shows. The analogue of this result in the context of crossed product C^* -algebras was open at the time this paper was submitted.

Finally, in Section 3.7 we show equivalence between primeness of the crossed product and topological transitivity of the induced system, in the case when A is unital and has an infinite character space. The analogue of this in the context of crossed product C^* -algebras is [7, Theorem 5.5].

3.2. Definition and a basic result

Let A be an associative commutative complex algebra and let $\Psi : A \rightarrow A$ be an algebra automorphism. Consider the set

$$A \rtimes_{\Psi} \mathbb{Z} = \{f : \mathbb{Z} \rightarrow A \mid f(n) = 0 \text{ except for a finite number of } n\}.$$

We endow it with the structure of an associative complex algebra by defining scalar multiplication and addition as the usual pointwise operations. Multiplication is defined by *twisted convolution*, $*$, as follows;

$$(f * g)(n) = \sum_{k \in \mathbb{Z}} f(k) \cdot \Psi^k(g(n-k)),$$

where Ψ^k denotes the k -fold composition of Ψ with itself. It is trivially verified that $A \rtimes_{\Psi} \mathbb{Z}$ is an associative \mathbb{C} -algebra under these operations. We call it the *crossed product of A by \mathbb{Z} under Ψ* .

A useful way of working with $A \rtimes_{\Psi} \mathbb{Z}$ is to write elements $f, g \in A \rtimes_{\Psi} \mathbb{Z}$ in the form $f = \sum_{n \in \mathbb{Z}} f_n \delta^n$, $g = \sum_{m \in \mathbb{Z}} g_m \delta^m$, where $f_n = f(n)$, $g_m = g(m)$, addition and scalar multiplication are canonically defined, and multiplication is determined by $(f_n \delta^n) * (g_m \delta^m) = f_n \cdot \Psi^n(g_m) \delta^{n+m}$, where $n, m \in \mathbb{Z}$ and $f_n, g_m \in A$ are arbitrary.

Clearly one may canonically view A as an abelian subalgebra of $A \rtimes_{\Psi} \mathbb{Z}$, namely as $\{f_0 \delta^0 \mid f_0 \in A\}$. The following elementary result is proved in [5, Proposition 2.1].

Proposition 3.2.1. *The commutant A' of A is abelian, and thus it is the unique maximal abelian subalgebra containing A .*

3.3. Setup and two basic results

In what follows, we shall focus on cases when A is a commutative complex Banach algebra, and freely make use of the basic theory for such A , see e.g. [2]. As conventions tend to differ slightly in the literature, however, we mention that we call a commutative Banach algebra A *semi-simple* if the Gelfand transform on A is injective, and that we call it *completely regular* (the term *regular* is also frequently used in the literature) if, for every subset $F \subseteq \Delta(A)$, where $\Delta(A)$ is the character space of A , that is closed in the Gelfand topology and for every $\phi_0 \in \Delta(A) \setminus F$, there exists an $a \in A$ such that $\widehat{a}(\phi) = 0$ for all $\phi \in F$ and $\widehat{a}(\phi_0) \neq 0$. All topological considerations of $\Delta(A)$ will be done with respect to its Gelfand topology (the weakest topology making all elements in the image of the Gelfand transform of A continuous on $\Delta(A)$).

Now let A be a complex commutative semi-simple completely regular Banach algebra, and let $\sigma : A \rightarrow A$ be an algebra automorphism. As in [5], σ induces a map $\tilde{\sigma} : \Delta(A) \rightarrow \Delta(A)$ defined by $\tilde{\sigma}(\mu) = \mu \circ \sigma^{-1}$, $\mu \in \Delta(A)$, which is automatically a homeomorphism when $\Delta(A)$ is endowed with the Gelfand topology. Hence we obtain a topological dynamical system $(\Delta(A), \tilde{\sigma})$. In turn, $\tilde{\sigma}$ induces an automorphism $\hat{\sigma} : \hat{A} \rightarrow \hat{A}$ (where \hat{A} denotes the algebra of Gelfand transforms of all elements of A) defined by $\hat{\sigma}(\hat{a}) = \hat{a} \circ \tilde{\sigma}^{-1} = \widehat{\sigma(a)}$. Therefore we can form the crossed product $\hat{A} \rtimes_{\hat{\sigma}} \mathbb{Z}$. We also mention that when speaking of ideals, we will always mean two-sided ideals.

In what follows, we shall make frequent use of the following fact. Its proof consists of a trivial direct verification.

Theorem 3.3.1. *Let A be a commutative semi-simple Banach algebra and σ an automorphism, inducing an automorphism $\hat{\sigma} : \hat{A} \rightarrow \hat{A}$ as above. Then the map $\Phi : A \rtimes_{\sigma} \mathbb{Z} \rightarrow \hat{A} \rtimes_{\hat{\sigma}} \mathbb{Z}$ defined by $\sum_{n \in \mathbb{Z}} a_n \delta^n \mapsto \sum_{n \in \mathbb{Z}} \hat{a}_n \delta^n$ is an isomorphism of algebras mapping A onto \hat{A} .*

Before stating the next result, we make the following basic definitions.

Definition 3.3.2. For any nonzero $n \in \mathbb{Z}$ we set

$$\text{Per}^n(\Delta(A)) = \{\mu \in \Delta(A) \mid \mu = \tilde{\sigma}^n(\mu)\}.$$

Furthermore, we denote the aperiodic points by

$$\text{Per}^{\infty}(\Delta(A)) = \bigcap_{n \in \mathbb{Z} \setminus \{0\}} (\Delta(A) \setminus \text{Per}^n(\Delta(A))).$$

Finally, for $f \in \hat{A}$, put

$$\text{supp}(f) = \overline{\{\mu \in \Delta(A) \mid f(\mu) \neq 0\}}.$$

We can now give the following explicit description of \hat{A}' in $\hat{A} \rtimes_{\hat{\sigma}} \mathbb{Z}$.

Theorem 3.3.3.

$$\hat{A}' = \left\{ \sum_{n \in \mathbb{Z}} f_n \delta^n \mid f_n \in \hat{A}, \text{ and for all } n \in \mathbb{Z} : \text{supp}(f_n) \subseteq \text{Per}^n(\Delta(A)) \right\}.$$

Proof. This follows from [5, Corollary 3.4], as \hat{A} trivially separates the points of $\Delta(A)$ and $\text{Per}^n(\Delta(A))$ is a closed set. \square

3.4. Three equivalent properties

In this section we shall conclude that, for certain A , two different algebraic properties of $A \rtimes_{\sigma} \mathbb{Z}$ are equivalent to density of the aperiodic points of the naturally associated dynamical system on the character space $\Delta(A)$, and hence obtain equivalence of three different

properties. The analogue of this result in the context of crossed product C^* -algebras is [7, Theorem 5.4]. We shall also combine this with a theorem from [5] to conclude a stronger result for the Banach algebra $L_1(G)$, where G is a locally compact abelian group with connected dual group.

In [5, Theorem 4.8], the following result is proved.

Theorem 3.4.1. *Let A be a complex commutative completely regular semi-simple Banach algebra, $\sigma : A \rightarrow A$ an automorphism and $\tilde{\sigma}$ the homeomorphism of $\Delta(A)$ in the Gelfand topology induced by σ as described above. Then the aperiodic points are dense in $\Delta(A)$ if and only if \widehat{A} is a maximal abelian subalgebra of $\widehat{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$. In particular, A is maximal abelian in $A \rtimes_{\sigma} \mathbb{Z}$ if and only if the aperiodic points are dense in $\Delta(A)$.*

We shall soon prove another algebraic property of the crossed product equivalent to density of the aperiodic points of the induced system on the character space. First, however, we need two easy topological lemmas.

Lemma 3.4.2. *Let $x \in \Delta(A)$ be such that the points $\tilde{\sigma}^i(x)$ are distinct for all i such that $-m \leq i \leq n$, where n and m are positive integers. Then there exist an open set U_x containing x such that the sets $\tilde{\sigma}^i(U_x)$ are pairwise disjoint for all i such that $-m \leq i \leq n$.*

Proof. It is easily checked that any finite set of points in a Hausdorff space can be separated by pairwise disjoint open sets. Separate the points $\tilde{\sigma}^i(x)$ with disjoint open sets V_i . Then it is readily verified that the set

$$U_x := \tilde{\sigma}^m(V_{-m}) \cap \tilde{\sigma}^{m-1}(V_{-m+1}) \cap \dots \cap V_0 \cap \tilde{\sigma}^{-1}(V_1) \cap \dots \cap \tilde{\sigma}^{-n}(V_n)$$

is an open neighbourhood of x with the required property. \square

Lemma 3.4.3. *The aperiodic points of $(\Delta(A), \tilde{\sigma})$ are dense if and only if $\text{Per}^n(\Delta(A))$ has empty interior for all positive integers n .*

Proof. Clearly, if there is a positive integer n_0 such that $\text{Per}^{n_0}(\Delta(A))$ has non-empty interior, the aperiodic points are not dense. For the converse, we recall that $\Delta(A)$ is a Baire space since it is locally compact and Hausdorff, and note that we may write

$$\Delta(A) \setminus \text{Per}^\infty(\Delta(A)) = \bigcup_{n>0} \text{Per}^n(\Delta(A)).$$

If the set of aperiodic points is not dense, its complement has non-empty interior, and as the sets $\text{Per}^n(\Delta(A))$ are clearly all closed, there must exist an integer $n_0 > 0$ such that $\text{Per}^{n_0}(\Delta(A))$ has non-empty interior since $\Delta(A)$ is a Baire space. \square

We are now ready to prove the promised result.

Theorem 3.4.4. *Let A be a complex commutative semi-simple completely regular Banach algebra, $\sigma : A \rightarrow A$ an automorphism and $\tilde{\sigma}$ the homeomorphism of $\Delta(A)$ in the Gelfand topology induced by σ as described above. Then the aperiodic points are dense in $\Delta(A)$ if and only if every non-zero ideal $I \subseteq A \rtimes_{\sigma} \mathbb{Z}$ is such that $I \cap A \neq \{0\}$.*

Proof. We first assume that $\overline{\text{Per}^\infty(\Delta(A))} = \Delta(A)$, and work initially in $\widehat{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$. Assume that $I \subseteq \widehat{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$ is a non-zero ideal, and that $f = \sum_{n \in \mathbb{Z}} f_n \delta^n \in I$. By definition, only finitely many f_n are non-zero. Denote the set of integers n for which $f_n \neq 0$ by $S = \{n_1, \dots, n_r\}$. Pick an aperiodic point $x \in \Delta(A)$ such that $f_{n_1}(x) \neq 0$ (by density of $\text{Per}^\infty(\Delta(A))$ such x exists). Using the fact that x is not periodic we may, by Lemma 3.4.2, choose an open neighbourhood U_x of x such that $\tilde{\sigma}^{-n_i}(U_x) \cap \tilde{\sigma}^{-n_j}(U_x) = \emptyset$ for $n_i \neq n_j$, $n_i, n_j \in S$. Now by complete regularity of A we can find a function $g \in \widehat{A}$ that is non-zero in $\tilde{\sigma}^{-n_1}(x)$, and vanishes outside $\tilde{\sigma}^{-n_1}(U_x)$. Consider $f * g = \sum_{n \in \mathbb{Z}} f_n \cdot (g \circ \tilde{\sigma}^{-n}) \delta^n$. This is an element in I and clearly the coefficient of δ^{n_1} is the only one that does not vanish on the open set U_x . Again by complete regularity of A , there is an $h \in \widehat{A}$ that is non-zero in x and vanishes outside U_x . Clearly $h * f * g = [h \cdot (g \circ \tilde{\sigma}^{-n_1}) f_{n_1}] \delta^{n_1}$ is a non-zero monomial belonging to I . Now any ideal that contains a non-zero monomial automatically contains a non-zero element of \widehat{A} . Namely, if $a_i \delta^i \in I$ then $[a_i \delta^i] * [(a_i \circ \tilde{\sigma}^i) \delta^{-i}] = a_i^2 \in \widehat{A}$. By the canonical isomorphism in Theorem 3.3.1, the result holds for $A \rtimes_{\sigma} \mathbb{Z}$ as well.

For the converse, assume that $\overline{\text{Per}^\infty(\Delta(A))} \neq \Delta(A)$. Again we work in $\widehat{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$. It follows from Lemma 3.4.3 that since $\overline{\text{Per}^\infty(\Delta(A))} \neq \Delta(A)$, there exists an integer $n > 0$ such that $\text{Per}^n(\Delta(A))$ has non-empty interior. As A is assumed to be completely regular, there exists $f \in \widehat{A}$ such that $\text{supp}(f) \subseteq \text{Per}^n(\Delta(A))$. Consider now the ideal $I = (f + f \delta^n)$. Using that f vanishes outside $\text{Per}^n(\Delta(A))$, we may rewrite this as follows

$$\begin{aligned} a_i \delta^i (f + f \delta^n) a_j \delta^j &= [a_i \cdot (a_j \circ \tilde{\sigma}^{-i}) \delta^i] * [f \delta^j + f \delta^{n+j}] \\ &= [a_i \cdot (a_j \circ \tilde{\sigma}^{-i}) \cdot (f \circ \tilde{\sigma}^{-i})] \delta^{i+j} + [a_i \cdot (a_j \circ \tilde{\sigma}^{-i}) \cdot (f \circ \tilde{\sigma}^{-i})] \delta^{i+j+n}. \end{aligned}$$

This means that any element in I may be written in the form $\sum_i (b_i \delta^i + b_i \delta^{n+i})$. As i runs only through a finite subset of \mathbb{Z} , this is not a non-zero monomial. In particular, it is not a non-zero element in \widehat{A} . Hence I intersects \widehat{A} trivially. By the canonical isomorphism in Theorem 3.3.1, the result carries over to $A \rtimes_{\sigma} \mathbb{Z}$. \square

Combining Theorem 3.4.1 and Theorem 3.4.4, we now have the following result.

Corollary 3.4.5. *Let A be a complex commutative semi-simple completely regular Banach algebra, $\sigma : A \rightarrow A$ an automorphism and $\tilde{\sigma}$ the homeomorphism of $\Delta(A)$ in the Gelfand topology induced by σ as described above. Then the following three properties are equivalent:*

- *The aperiodic points $\text{Per}^\infty(\Delta(A))$ of $(\Delta(A), \tilde{\sigma})$ are dense in $\Delta(A)$;*
- *Every non-zero ideal $I \subseteq A \rtimes_{\sigma} \mathbb{Z}$ is such that $I \cap A \neq \{0\}$;*
- *A is a maximal abelian subalgebra of $A \rtimes_{\sigma} \mathbb{Z}$.*

We shall make use of Corollary 3.4.5 to conclude a result for a more specific class of Banach algebras. We start by recalling a number of standard results from the theory of Fourier analysis on groups, and refer to [2] and [4] for details. Let G be a locally compact abelian group. Recall that $L_1(G)$ consists of equivalence classes of complex-valued Borel measurable functions of G that are integrable with respect to a Haar measure on G , and that

$L_1(G)$ equipped with convolution product is a commutative completely regular semi-simple Banach algebra. A group homomorphism $\gamma : G \rightarrow \mathbb{T}$ from a locally compact abelian group to the unit circle is called a *character* of G . The set of all *continuous* characters of G forms a group Γ , the *dual group* of G , if the group operation is defined by

$$(\gamma_1 + \gamma_2)(x) = \gamma_1(x)\gamma_2(x) \quad (x \in G; \gamma_1, \gamma_2 \in \Gamma).$$

If $\gamma \in \Gamma$ and if

$$\widehat{f}(\gamma) = \int_G f(x)\gamma(-x)dx \quad (f \in L_1(G)),$$

then the map $f \mapsto \widehat{f}(\gamma)$ is a non-zero complex homomorphism of $L_1(G)$. Conversely, every non-zero complex homomorphism of $L_1(G)$ is obtained in this way, and distinct characters induce distinct homomorphisms. Thus we may identify Γ with $\Delta(L_1(G))$. The function $\widehat{f} : \Gamma \rightarrow \mathbb{C}$ defined as above is called the *Fourier transform* of $f \in L_1(G)$, and is precisely the Gelfand transform of f . We denote the set of all such \widehat{f} by $A(\Gamma)$. Furthermore, Γ is a locally compact abelian group in the Gelfand topology.

In [5, Theorem 4.16], the following result is proved.

Theorem 3.4.6. *Let G be a locally compact abelian group with connected dual group and let $\sigma : L_1(G) \rightarrow L_1(G)$ be an automorphism. Then $L_1(G)$ is maximal abelian in $L_1(G) \rtimes_{\sigma} \mathbb{Z}$ if and only if σ is not of finite order.*

Combining Corollary 3.4.5 and Theorem 3.4.6 the following result is immediate.

Corollary 3.4.7. *Let G be a locally compact abelian group with connected dual group and let $\sigma : L_1(G) \rightarrow L_1(G)$ be an automorphism. Then the following three statements are equivalent.*

- σ is not of finite order;
- Every non-zero ideal $I \subseteq L_1(G) \rtimes_{\sigma} \mathbb{Z}$ is such that $I \cap L_1(G) \neq \{0\}$;
- $L_1(G)$ is a maximal abelian subalgebra of $L_1(G) \rtimes_{\sigma} \mathbb{Z}$.

3.5. Minimality versus simplicity

Recall that a topological dynamical system is said to be *minimal* if all of its orbits are dense, and that an algebra is called *simple* if it lacks non-trivial proper ideals.

Theorem 3.5.1. *Let A be a complex commutative semi-simple completely regular unital Banach algebra such that $\Delta(A)$ consists of infinitely many points, and let $\sigma : A \rightarrow A$ be an algebra automorphism of A . Then $A \rtimes_{\sigma} \mathbb{Z}$ is simple if and only if the naturally induced system $(\Delta(A), \widetilde{\sigma})$ is minimal.*

Proof. Suppose first that the system is minimal, and assume that I a proper ideal of $A \rtimes_{\sigma} \mathbb{Z}$. Note that $I \cap A$ is a proper σ - and σ^{-1} -invariant ideal of A . By basic theory of Banach algebras, $I \cap A$ is contained in a maximal ideal of A (note that $I \cap A \neq A$ as A is unital

and I was assumed to be proper), which is the kernel of an element $\mu \in \Delta(A)$. Now $\widehat{I \cap A}$ is a $\widehat{\sigma}$ - and $\widehat{\sigma}^{-1}$ -invariant proper non-trivial ideal of \widehat{A} , all of whose elements vanish in μ . Invariance of this ideal implies that all of its elements even annihilate the whole orbit of μ under $\widehat{\sigma}$. But by minimality, every such orbit is dense and hence $\widehat{I \cap A} = \{0\}$. By semi-simplicity of A , this means $I \cap A = \{0\}$, so $I = \{0\}$ by Corollary 3.4.5. For the converse, assume that there is an element $\mu \in \Delta(A)$ whose orbit $O(\mu)$ is not dense. By complete regularity of A there is a nonzero $g \in \widehat{A}$ that vanishes on $\overline{O(\mu)}$. Then clearly the ideal generated by g in $\widehat{A} \rtimes_{\widehat{\sigma}} \mathbb{Z}$ consists of finite sums of elements of the form $(f_n \delta^n) * g * (h_m \delta^m) = [f_n \cdot (g \circ \widehat{\sigma}^{-n}) \cdot (h_m \circ \widehat{\sigma}^{-n})] \delta^{n+m}$, and hence the coefficient of every power of δ in this ideal must vanish in μ , whence the ideal is proper. Hence by Theorem 3.3.1, $A \rtimes_{\sigma} \mathbb{Z}$ is not simple. \square

3.6. Every non-zero ideal has non-zero intersection with A'

We shall now show that any non-zero ideal of $A \rtimes_{\sigma} \mathbb{Z}$ has non-zero intersection with A' . This should be compared with Corollary 3.4.5, which says that a non-zero ideal may well intersect A solely in 0. There was no known analogue of this result in the context of crossed product C^* -algebras in the literature at the time this paper was submitted.

Theorem 3.6.1. *Let A be a complex commutative semi-simple completely regular Banach algebra, and $\sigma : A \rightarrow A$ an automorphism. Then every non-zero ideal I in $A \rtimes_{\sigma} \mathbb{Z}$ has non-zero intersection with the commutant A' of A , that is $I \cap A' \neq \{0\}$.*

Proof. As usual, we work in $\widehat{A} \rtimes_{\widehat{\sigma}} \mathbb{Z}$. When $\overline{\text{Per}^{\infty}(\Delta(A))} = \Delta(A)$, the result follows immediately from Corollary 3.4.5. We will use induction on the number of non-zero terms in an element $f = \sum_{n \in \mathbb{Z}} f_n \delta^n$ to show that it generates an ideal that intersects \widehat{A}' non-trivially. The starting point for the induction, namely when $f = f_n \delta^n$ with non-zero f_n , is clear since any such element generates an ideal that even intersects \widehat{A} non-trivially, as was shown in the proof of Theorem 3.4.4. Now assume inductively that the conclusion of the theorem is true for the ideals generated by any element of $\widehat{A} \rtimes_{\widehat{\sigma}} \mathbb{Z}$ with r non-zero terms for some positive integer r , and consider an element $f = f_{n_1} \delta^{n_1} + \dots + f_{n_{r+1}} \delta^{n_{r+1}}$. By multiplying from the right with a suitable element we obtain an element in the ideal generated by f of the form $g = \sum_{i=0}^{m_r} g_i \delta^i$ such that $g_0 \neq 0$. If some of the other g_i are zero we are done by induction hypothesis, so we may assume this is not the case. We may also assume that g is not in the commutant of \widehat{A} since otherwise we are of course also done. This means, by Theorem 3.3.3, that there is such j that $0 < j \leq m_r$ and $\text{supp}(g_j) \not\subseteq \text{Per}^j(\Delta(A))$. Pick an $x \in \text{supp}(g_j)$ such that $x \neq \widehat{\sigma}^{-j}(x)$ and $g_j(x) \neq 0$. As $\Delta(A)$ is Hausdorff we can choose an open neighbourhood U_x of x such that $U_x \cap \widehat{\sigma}^{-j}(U_x) = \emptyset$. Complete regularity of A implies existence of an $h \in \widehat{A}$ such that $h \circ \widehat{\sigma}^{-j}(x) = 1$ and h vanishes identically outside of $\widehat{\sigma}^{-j}(U_x)$. Now $g * h = \sum_{i=0}^{m_r} g_i \cdot (h \circ \widehat{\sigma}^{-i}) \delta^i$. Using complete regularity of A again we pick a function $a \in \widehat{A}$ such that $a(x) = 1$ and a vanishes outside U_x . We have $a * g * h = \sum_{i=0}^{m_r} a \cdot g_i \cdot (h \circ \widehat{\sigma}^{-i}) \delta^i$, which is in the ideal generated by f . Now $a \cdot g_0 \cdot h$ is identically zero since $a \cdot h = 0$. On the other hand, $a \cdot g_j \cdot (h \circ \widehat{\sigma}^{-j})$ is non-zero in the point x . Hence $a * g * h$ is a non-zero element in the ideal generated by f whose number of non-zero coefficient functions is less than or equal to r . By the induction hypothesis, such an element

generates an ideal that intersects the commutant of \widehat{A} non-trivially. By Theorem 3.3.1 it follows that every non-zero ideal of $A \rtimes_{\sigma} \mathbb{Z}$ intersects A' non-trivially. \square

3.7. Primeness versus topological transitivity

We shall show that for certain A , $A \rtimes_{\sigma} \mathbb{Z}$ is prime if and only if the induced system $(\Delta(A), \tilde{\sigma})$ is topologically transitive. The analogue of this result in the context of crossed product C^* -algebras is in [7, Theorem 5.5].

Definition 3.7.1. The system $(\Delta(A), \tilde{\sigma})$ is called *topologically transitive* if for any pair of non-empty open sets U, V of $\Delta(A)$, there exists an integer n such that $\tilde{\sigma}^n(U) \cap V \neq \emptyset$.

Definition 3.7.2. The algebra $A \rtimes_{\sigma} \mathbb{Z}$ is called *prime* if the intersection between any two non-zero ideals I, J is non-zero, that is $I \cap J \neq \{0\}$.

For convenience, we also make the following definition.

Definition 3.7.3. The map $E : \widehat{A} \rtimes_{\tilde{\sigma}} \mathbb{Z} \rightarrow \widehat{A}$ is defined by $E(\sum_{n \in \mathbb{Z}} f_n \delta^n) = f_0$.

To prove the main theorem of this section, we need the two following topological lemmas.

Lemma 3.7.4. *If $(\Delta(A), \tilde{\sigma})$ is not topologically transitive, then there exist two disjoint invariant non-empty open sets O_1 and O_2 such that $\overline{O_1} \cup \overline{O_2} = \Delta(A)$.*

Proof. As the system is not topologically transitive, there exist non-empty open sets $U, V \subseteq \Delta(A)$ such that for any integer n we have $\tilde{\sigma}^n(U) \cap V = \emptyset$. Now clearly the set $O_1 = \bigcup_{n \in \mathbb{Z}} \tilde{\sigma}^n(U)$ is an invariant non-empty open set. Then $\overline{O_1}$ is an invariant closed set. It follows that $O_2 = \Delta(A) \setminus \overline{O_1}$ is an invariant open set containing V . Thus we even have that $\overline{O_1} \cup O_2 = \Delta(A)$, and the result follows. \square

Lemma 3.7.5. *If $(\Delta(A), \tilde{\sigma})$ is topologically transitive and there is an $n_0 > 0$ such that $\Delta(A) = \text{Per}^{n_0}(\Delta(A))$, then $\Delta(A)$ consists of a single orbit and is thus finite.*

Proof. Assume two points $x, y \in \Delta(A)$ are not in the same orbit. As $\Delta(A)$ is Hausdorff we may separate the points $x, \sigma(x), \dots, \sigma^{n_0-1}(x), y$ by pairwise disjoint open sets $V_0, V_1, \dots, V_{n_0-1}, V_y$. Now consider the set

$$U_x := V_0 \cap \tilde{\sigma}^{-1}(V_1) \cap \tilde{\sigma}^{-2}(V_2) \cap \dots \cap \tilde{\sigma}^{-n_0+1}(V_{n_0-1}).$$

Clearly the sets $A_x = \bigcup_{i=0}^{n_0-1} \tilde{\sigma}^i(U_x)$ and $A_y = \bigcup_{i=0}^{n_0-1} \tilde{\sigma}^i(V_y)$ are disjoint invariant non-empty open sets, which leads us to a contradiction. Hence $\Delta(A)$ consists of one single orbit under $\tilde{\sigma}$. \square

We are now ready for a proof of the following result.

Theorem 3.7.6. *Let A be a complex commutative semi-simple completely regular unital Banach algebra such that $\Delta(A)$ consists of infinitely many points, and let σ be an automorphism of A . Then $A \rtimes_{\sigma} \mathbb{Z}$ is prime if and only if the associated system $(\Delta(A), \tilde{\sigma})$ on the character space is topologically transitive.*

Proof. Suppose that the system $(\Delta(A), \tilde{\sigma})$ is not topologically transitive. Then there exists, by Lemma 3.7.4, two disjoint invariant non-empty open sets O_1 and O_2 such that $\overline{O_1} \cup \overline{O_2} = \Delta(A)$. Let I_1 and I_2 be the ideals generated in $\widehat{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$ by $k(\overline{O_1})$ (the set of all functions in \widehat{A} that vanish on $\overline{O_1}$) and $k(\overline{O_2})$ respectively. We have that

$$E(I_1 \cap I_2) \subseteq E(I_1) \cap E(I_2) = k(\overline{O_1}) \cap k(\overline{O_2}) = k(\overline{O_1} \cup \overline{O_2}) = k(\Delta(A)) = \{0\}.$$

It is not difficult to see that if $I \subseteq \widehat{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$ is an ideal and $E(I) = \{0\}$, then $I = \{0\}$. Namely, suppose $F = \sum_{n \in \mathbb{Z}} f_n \delta^n \in I$ and $f_i \neq 0$ for some integer i . Since A is unital, so is \widehat{A} and thus $\delta^{-i} \in \widehat{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$. So $F * \delta^{-i} \in I$ and hence $E(F * \delta^{-i}) = f_i = 0$ which is a contradiction, so $I = \{0\}$. Hence $I_1 \cap I_2 = \{0\}$ and $\widehat{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$ is not prime. By Theorem 3.3.1, neither is $A \rtimes_{\sigma} \mathbb{Z}$. Next suppose that $(\Delta(A), \tilde{\sigma})$ is topologically transitive. Assume that $\text{Per}^{\infty}(\Delta(A))$ is not dense. Then by Lemma 3.4.3 there is an integer $n_0 > 0$ such that $\text{Per}^{n_0}(\Delta(A))$ has non-empty interior. As $\text{Per}^{n_0}(\Delta(A))$ is invariant and closed, topological transitivity implies that $\Delta(A) = \text{Per}^{n_0}(\Delta(A))$. This, however, is impossible since by Lemma 3.7.5 it would force $\Delta(A)$ to consist of a single orbit and hence be finite. Thus $\text{Per}^{\infty}(\Delta(A))$ is dense after all. Now let I and J be two non-zero proper ideals in $A \rtimes_{\sigma} \mathbb{Z}$. Unitality of A assures us that $I \cap A$ and $J \cap A$ are proper σ - and σ^{-1} -invariant ideals of A and density of $\text{Per}^{\infty}(\Delta(A))$ assures us that they are non-zero, by Corollary 3.4.5. Consider $A_I = \{\mu \in \Delta(A) \mid \mu(a) = 0 \text{ for all } a \in I \cap A\}$ and $A_J = \{\nu \in \Delta(A) \mid \nu(b) = 0 \text{ for all } b \in J \cap A\}$. Now by Banach algebra theory a proper ideal of a commutative unital Banach algebra A is contained in a maximal ideal, and a maximal ideal of A is always precisely the set of zeroes of some $\zeta \in \Delta(A)$. This implies that both A_I and A_J are non-empty, and semi-simplicity of A assures us that they are proper subsets of $\Delta(A)$. They are clearly also closed and invariant under $\tilde{\sigma}$ and $\tilde{\sigma}^{-1}$. Hence $\Delta(A) \setminus A_I$ and $\Delta(A) \setminus A_J$ are invariant non-empty open sets. By topological transitivity we must have that these two sets intersect, hence that $A_I \cup A_J \neq \Delta(A)$. This means that there exists $\eta \in \Delta(A)$ and $a \in I \cap A$, $b \in J \cap A$ such that $\eta(a) \neq 0$, $\eta(b) \neq 0$ and hence that $\eta(ab) \neq 0$. Hence $0 \neq ab \in I \cap J$, and we conclude that $A \rtimes_{\sigma} \mathbb{Z}$ is prime. \square

Acknowledgments

This work was supported by a visitor's grant of the *Netherlands Organisation for Scientific Research (NWO)*, *The Swedish Foundation for International Cooperation in Research and Higher Education (STINT)*, *Crafoord Foundation* and *The Royal Physiographic Society in Lund*.

We are also grateful to Jun Tomiyama and Johan Öinert for useful discussions.

References

- [1] Davidson, K.R., *C*-algebras by example*, Fields Institute Monographs no. 6, Amer. Math. Soc., Providence RI, 1996.
- [2] Larsen, R., *Banach algebras: an introduction*, Marcel Dekker, Inc., New York, 1973.

-
- [3] Power, S.C., *Simplicity of C^* -algebras of minimal dynamical systems*, J. London Math. Soc. **18** (1978), 534-538.
 - [4] Rudin, W., *Fourier analysis on groups*, Interscience Publishers, New York, London, 1962.
 - [5] Svensson, C., Silvestrov, S., de Jeu, M., *Dynamical systems and commutants in crossed products*, Internat. J. Math. **18** (2007), 455-471.
 - [6] Tomiyama, J., *Invitation to C^* -algebras and topological dynamics*, World Sci., Singapore, New Jersey, Hong Kong, 1987.
 - [7] Tomiyama, J., *The interplay between topological dynamics and theory of C^* -algebras*, Lecture Note no.2, Global Anal. Research Center, Seoul, 1992.
 - [8] Williams, D.P., *Crossed products of C^* -algebras*, Mathematical Surveys and Monographs no. 134, American Mathematical Society, Providence RI, 2007.

