Chapter 1

Introduction

The theoretical foundation for the work reported on in this thesis is provided by the scattering theory of electron transport. When a conductor is small enough, electrons can traverse it without undergoing inelastic processes. (Inelastic processes include, among other things, collisions between electrons and interactions with phonons or photons.) Transport is then said to be coherent. Such small conductors (typical lengths are between a few nanometers and a few microns) can be thought of as electron wave guides. At a given energy, transport takes place through a discrete set of channels. Transport properties are characterized by a scattering matrix. This insight is due to Landauer [1].

![Figure 1.1. A one-dimensional scattering problem: A wave (2) impinges on a scattering center (3). As a result, a fraction $T$ of the incident wave intensity is transmitted and a fraction $R = 1 - T$ is reflected. The conductance $G$ of the scatterer equals $(e^2/h)T$.](image)

In Fig. 1.1 the three main ingredients of a scattering problem are schematically indicated. These are (1) a set of reservoirs that emit and absorb particles, (2) the particles themselves, that propagate as waves between the reservoirs and (3) a scatterer that obstructs the free propagation
of waves.

In this thesis, we consider two classes of problems. The first class results when the physical quantities that characterize the reservoirs or those that characterize the scatterer are not constant in time. The time-dependence can have several causes: (1) An experimentalist pressed a button or turned a nob and changed the value of a system parameter such as the external bias voltage or the magnetic field. (2) The internal dynamics of the reservoirs or scatterer causes them to evolve significantly on time-scales that are of the same order as those associated with the electrons being transported. (3) The state of a reservoir or of the scatterer changes due to interactions with the electrons being transported. Situations of this kind are considered in Chapters 2, 3 and 4. The second class of problems results when wave propagation is described by the Dirac equation rather than the Schrödinger equation. The Dirac equation describes excitations in a recently discovered form of carbon called graphene, and forms the basis of Chapters 5, 6, and 7.

In this introductory chapter we give some background to the main topics and methods of this thesis.

1.1 Scattering theory of electron transport

We start by briefly discussing the scattering theory of electron transport, known as the Landauer-Büttiker formalism [1, 2]. (For a more detailed review, see [3].)

A coherent conductor such as the one in Fig. 1.2 is characterized by a scattering matrix. For the purpose of introducing the scattering matrix, we consider a two terminal device, but note that the theory readily generalizes to more terminals. We denote the two terminals left and right. The scattering matrix $s(E)$ is defined at a given energy $E$. Without loss of generality we may take the number $N$ of channels for left-incident electrons to be the same as the number of channels for right-incident electrons. Then $s(E)$ is a $2N \times 2N$ matrix with four $N \times N$ sub-blocks

$$ s(E) = \begin{pmatrix} r(E) & t'(E) \\ t(E) & r'(E) \end{pmatrix}. $$

(1.1)

The entries of these sub-blocks are reflection and transmission amplitudes. An electron with energy $E$ that enters the conductor at the left terminal in channel $n$ has an amplitude $r_{mn}(E)$ to be reflected back into the left
1.1 Scattering theory of electron transport

Figure 1.2. One of the very first fabricated coherent conductors. The figure shows a scanning electron micrograph of a double quantum point contact. The bright regions are electrostatic gate electrodes. They sit on top of a GaAs-AlGaAs heterostructure. The heterostructure contains a two dimensional electron gas. The electrodes deplete charge in the regions underneath them, thus defining a barrier with two small openings or quantum point contacts. The white lines at the bottom of the figure denote a length of $1 \mu m$. (From van Houten et al [4].)

terminal in channel $m$ and an amplitude $t_{mn}(E)$ to be transmitted into the right terminal in channel $m$. Similarly, an electron that enters at the right terminal in channel $n$ has an amplitude $r'_{mn}(E)$ to be reflected back into the right terminal in channel $m$ and an amplitude $t'_{mn}(E)$ to be transmitted into the left terminal in channel $m$.

Transport properties are expressed in terms of the eigenvalues $T_n(E)$ of the transmission matrix product

$$T(E) = t(E)t(E)^\dagger. \quad (1.2)$$

(Our choice of using the left to right transmission amplitudes is arbitrary. As a consequence of the unitarity of the scattering matrix, the nonzero eigenvalues of $tt^\dagger$ and $t't'^\dagger$ are identical.) The eigenvalues $T_n$ lie between zero and one.

Suppose now that the conductor is connected to two large reservoirs. In the left (+) and right (−) reservoirs, the electrons are described by Fermi distributions

$$f_\pm(E) = \frac{1}{1 + \exp[(E - E_F \pm eV/2)/k_BT]}, \quad (1.3)$$

where $e$ is the electron charge, $V$ is the voltage bias across the conductor, $E_F$ is the Fermi energy, $k_B$ is Boltzmann’s constant and $T$ is the
temperature. Let $I(t)$ be the Heisenberg operator for current through the conductor. The expectation value for the current through the conductor $\bar{I} = \langle I(t) \rangle$ is time-independent since there are no time-dependent external fields. A central result of the scattering theory is the Landauer formula, that relates $\bar{I}$ to $T_n$. The formula reads

$$I = \frac{e}{2\pi\hbar} \sum_n \int dE T_n(E) [f_+(E) - f_-(E)].$$

Due to the difference between Fermi functions, the integrand is non-zero in an interval of a few times $\max\{eV, k_B T\}$ around the Fermi energy. This has to be compared to the energy scale at which $T_n(E)$ is constant. This is the Thouless energy $E_{Th}$. The low-temperature, small-voltage regime is defined by $eV, k_B T \ll E_{Th}$. (It must be noted that often great experimental effort is needed to enter this response regime.) In this regime, we may evaluate $T_n(E)$ at the Fermi energy $E_F$ and take it outside the integral. Using the fact that $\int dE \left[ f_+(E) - f_-(E) \right] = eV$ independent of temperature, we obtain $I = GV$ with the linear response conductance

$$G = \frac{e^2}{2\pi\hbar} \sum_n T_n.$$

Since $T_n \leq 1$, this result means that each transport channel $n$ contributes at most $G_Q = e^2/2\pi\hbar$ to the conductance.

The conductance only contains information about the time averaged current $\bar{I}$. The current fluctuations contain additional information. The fluctuations are characterized by the Fourier transform of the current-current correlation function, which is known as the power spectrum $P(\omega)$ of the current noise,

$$P(\omega) = 2 \int_{-\infty}^{\infty} dt \, e^{i\omega t} \left[ \langle I(t)I(0) \rangle - \bar{I}^2 \right].$$

The noise can also be expressed in terms of the transmission eigenvalues $T_n$. In the regime, $\omega \ll eV, k_B T \ll E_{Th}$, the general result is

$$P = \frac{e^2}{\pi\hbar} \left[ 2k_B T \sum_n T_n^2 + eV \coth \left( \frac{eV}{2k_B T} \right) \sum_n T_n (1 - T_n) \right].$$

There are two contributions to this result. The first is thermal, i.e. fluctuations induced by the finite temperature of the system. The second
1.2 The Keldysh technique

A system is in equilibrium when the Hamiltonian \( H \) that governs its dynamics is time-independent and the state of the system is characterized by a density matrix \( \rho = Z^{-1} e^{-\beta(H-\mu N)} \), with \( N \) the number of particles, \( \mu \) the chemical potential, and \( Z \) the partition function \( Z = \text{tr} e^{-\beta(H-\mu N)} \). For such systems, the equilibrium Green function technique can be used to calculate expectation values of observables [14]. The central object is the so-called retarded Green function and the technique provides a systematic procedure to compute it.
There are many situations that fall beyond the scope of equilibrium theory. For instance, a system can be prepared in a non-equilibrium initial state and its subsequent relaxation studied. An open system can be in contact with an environment that drives it away from equilibrium. This is the case when different reservoirs connected to the same system have different temperatures or chemical potentials. Non-equilibrium states also arise when time-dependent external fields are applied. In these situations it is no longer sufficient to know the retarded Green function only. More information is required in order to evaluate expectation values of observables. This information is contained in an additional Green function, called the Keldysh Green function. The Keldysh technique provides a procedure, formally similar to the one used in equilibrium theory, with which to compute all necessary Green functions.

We will consider electron systems, and therefore make definitions appropriate for fermions. The Keldysh Green function $K_{m,n}(t,t')$ and the retarded Green function $R_{m,n}(t,t')$ are defined as

\[
K_{m,n}(t,t') \equiv -i \langle a_m(t), a_n^\dagger(t') \rangle, \quad (1.10a)
\]

\[
R_{m,n}(t,t') \equiv -i \theta(t-t') \langle \left\{ a_m(t), a_n^\dagger(t') \right\} \rangle, \quad (1.10b)
\]

where $a_m^\dagger(t)$ is the Heisenberg-picture operator that creates an electron in a single-particle state $m$. While it does not provide new information, it is useful to define also the advanced Green function $A_{m,n}(t,t') \equiv R_{n,m}(t',t)^*$. The key result of Keldysh’s theory is this: The three Green functions constitute a matrix

\[
G_{m,n}(t,t') = \begin{pmatrix} R_{m,n}(t,t') & K_{m,n}(t,t') \\ 0 & A_{m,n}(t,t') \end{pmatrix}, \quad (1.11)
\]

that obeys two integro-differential equations that are formally similar to those known from equilibrium theory.

At this point it is usual to introduce the Keldysh time contour and contour ordered Green functions [10]. Since we will simply state rather than derive the integro-differential equations for $G$, we do not need to discuss contour ordering here. For completeness we only mention that $G_{m,n}(t,t')$ as we have defined it, is related to the contour ordered Green function $\hat{G}_{m,n}(t,t')$ as defined in Ref. [10] by the transformation $\hat{G}_{m,n}(t,t') = \tau_3 L^\dagger G_{m,n}(t,t') L$ where

\[
\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \quad (1.12)
\]
In order to find equations that determine \( G \), the system is split up into a single-particle problem that can be solved exactly and a perturbation. In the single-particle basis that diagonalizes the unperturbed system, the equations for \( G \) are of the form

\[
(i \partial_t - \varepsilon_m)G_{m,n}(t, t') - \sum_l \int d\tilde{t} \Sigma_{m,l}(t, \tilde{t}) G_{l,n}(\tilde{t}, t') = \delta(t - t')\delta_{m,n}I_{2\times2}, \tag{1.13a}
\]

\[
(-i \partial_{t'} - \varepsilon_n)G_{m,n}(t, t') - \sum_l \int d\tilde{t} G_{m,l}(t, \tilde{t}) \Sigma_{l,n}(\tilde{t}, t') = \delta(t - t')\delta_{m,n}I_{2\times2}, \tag{1.13b}
\]

where \( I_{2\times2} \) is the \( 2 \times 2 \) unit matrix and \( \varepsilon_m \) is the energy of level \( m \). The self-energy

\[
\Sigma_{m,n}(t, t') = \begin{pmatrix}
\Sigma_{m,n}^{(R)}(t, t') & \Sigma_{m,n}^{(K)}(t, t') \\
0 & \Sigma_{m,n}^{(A)}(t, t')
\end{pmatrix}
\]

(1.14)

takes account of the perturbation. The perturbation may for instance be due to electron-electron interactions, electron-phonon interactions or impurity scattering. (See Ref. [10] for details.) The self-energy can also take into account time dependent external fields. In Chapter 2 we use it to describe the presence of reservoirs. The formalism is able to deal with reservoirs characterized by arbitrary time-dependent distribution functions. As in equilibrium, the self-energy is the sum of all amputated one-particle-irreducible diagrams. (See for example Ref. [15] for a detailed exposition of equilibrium diagrammatics.) The only difference is that now the propagators have an additional \( 2 \times 2 \) matrix structure.

In equilibrium, the self-energy determines the effective single-particle spectrum. Outside equilibrium it also dictates how particles are distributed among the single-particle levels. The reader who wants to know more is referred to Appendix 1.A where we work through a simple example that employs the Keldysh technique.

### 1.3 Photons

In Sec. 1.1 we considered ideal reservoirs. As a result, the voltage across the conductor did not fluctuate. This was an idealization. A more realistic
Figure 1.3. Circuit that models fluctuations in the bias voltage across a coherent conductor with conductance $G$. The voltage fluctuations are produced by current fluctuations in the coherent conductor. The external circuit is represented by an impedance $Z$. The voltage in node $a$ develops fluctuations when $ZG > 1$.

model results when one connects an impedance $Z$ in series with the coherent conductor with conductance $G$ and then biases the combined structure with an ideal voltage [16, 17]. (See Fig. 1.3). The impedance represents the external circuit to which the coherent conductor is connected. The ideal bias voltage is divided between the impedance and the coherent conductor. This division is not constant. Fluctuations in the current through the coherent conductor are converted into voltage fluctuations in the shared node. The fluctuations become important when $ZG > 1$.

One way to describe these fluctuations is to treat the external circuit as a quantum system [18] in the same way that the coherent conductor is a quantum system. This was first done in the context of ultra small tunnel junctions [19, 20, 21]. Generalization to arbitrary coherent conductors was achieved in Refs. [16, 17, 22]. The construction is based on the theory of Caldeira and Leggett [23, 24] and exploits the fact that any impedance can be represented by means of a quadratic Hamiltonian for a set of bosonic modes. As a result, the description that emerges for a coherent conductor coupled to an external circuit is that of a quantum system in contact with a bosonic bath. The bosonic modes that couple to the coherent conductor are photons [18, 25, 26]. They quantize the energy stored in electromagnetic fields in the neighborhood of the coherent conductor.

Here we briefly discuss this quantization. We define a variable $\varphi$ such that the potential difference between the node and infinity (or earth) is
Figure 1.4. A two level system is placed in the vicinity of node $a$. The two level system consists of an electron that can hop between two localized states 1 and 2. The photons associated with voltage fluctuations in node $a$ interact with the electron.

$V = d\varphi/dt$. This potential difference is related to the charge $Q$ stored on the node by its capacitance $V = Q/C$. The energy stored in the resulting electric field is $H_C = Q^2/2C$ so that

$$\frac{d\varphi}{dt} = \frac{\partial H_C}{\partial Q}.$$  

(1.15)

This is a Hamilton equation of motion with $\varphi$ a generalized coordinate and $Q$ its conjugate momentum. Note that $H_C$ is not the full Hamiltonian of the circuit but that we assume it is the only term that contains $Q$.

Quantization of these degrees of freedom then follows the standard procedure. $Q$ and $\varphi$ are promoted to operators and the commutation relation

$$[\hat{\varphi}, \hat{Q}] = i\hbar$$

(1.16)

is imposed. This allows us to define operators $b$ and $b^\dagger$ through

$$b = \frac{\hat{Q}}{\sqrt{2\hbar C\omega}} - i\sqrt{C\omega/2\hbar} \hat{\varphi},$$

(1.17)

that obey the bosonic commutation relation $[b, b^\dagger] = 1$. The parameter $\omega$ is arbitrary at this point, but a natural choice is dictated by the dynamics of the circuit under consideration. The particles associated with $b$ and $b^\dagger$ are photons.

We must next consider the interaction of quantized radiation with other quantum systems. This interaction can be derived from the fundamental $j \cdot A$ coupling term of quantum electrodynamics [27]. A simple example
serves to illustrate the general principle. Consider therefore an electron that can hop between two localized, spatially separated states \(|1\rangle\) and \(|2\rangle\) in the vicinity of the node \(a\). For definiteness, let us say that when the electron is in state \(|1\rangle\) it is closer to node \(a\) than when it is in state \(|2\rangle\). (See Fig. 1.4). What happens when the electron is moved closer to or further from the metallic node \(a\)? The electron induces a charge on the node. (This charge screens the interior of the node from the electric field produced by the electron.) If the electron is far from the node, it produces only a small electric field at the node. In this case the induced charge on the node is also small. When the electron is closer to the node, the electric field it produces at the node is larger and so too is the charge it induces on the node. Thus we see that hopping of the electron between states \(|1\rangle\) and \(|2\rangle\) is accompanied by a change of the charge on the node \(a\). The charge on node \(a\) in turn produces its own electric field to which the electron is sensitive. We now develop a quantum mechanical description of this dynamics.

We start by looking at the electron in the absence of any electromagnetic fields when its Hamiltonian is

\[ H = \gamma |2\rangle \langle 1| + \gamma^* |1\rangle \langle 2|. \]

Here \(\gamma\) is the hopping amplitude from \(|1\rangle\) to \(|2\rangle\). How is the Hamiltonian modified in the presence of electromagnetic fields? First, let us consider classical fields. We also restrict ourselves to the regime where the electric field dominates, so that we can neglect the magnetic field by setting \(\nabla \times A = 0\). We choose the gauge in which the scalar potential is zero and the electric field is given in terms of the vector potential as

\[ E = -\dfrac{\partial A}{\partial t}. \quad (1.18) \]

In this gauge, the effect of the electric field is to modify the hopping amplitude

\[ \gamma \rightarrow \gamma \exp \left( \dfrac{ie}{\hbar} \int_{1\rightarrow 2} dl \cdot A \right), \quad (1.19) \]

where the line-integral runs along any path from site 1 to site 2.

The next step is to note that a charge \(Q\) on node \(a\) produces an electric field \(E\) proportional to \(Q\): If we double the charge, we double the electric field everywhere in space. From Eq. (1.18) then follows

\[ \dfrac{\partial}{\partial t} \int_{1\rightarrow 2} dl \cdot A = -\int_{1\rightarrow 2} dl \cdot E \propto Q = CV. \quad (1.20) \]
1.4 Graphene: Dirac Fermions

Since the voltage $V$ in node $a$ is related to the phase $\varphi$ through $d\varphi/dt = V$, it follows that

$$\int_{1\to 2} dl \cdot A = \alpha \varphi, \quad (1.21)$$

where $\alpha$ is a dimensionless proportionality constant. This equation allows us to go over to a quantum description. The classical phase $\varphi$ is replaced by the quantum mechanical operator $\hat{\varphi}$. If $H_c$ is the Hamiltonian for the circuit degrees of freedom, then the total Hamiltonian for the circuit plus electron is $H = H_c + H_e$, where

$$H_e = \gamma e^{i\alpha e \hat{\varphi}/\hbar} |2\rangle \langle 1| + \gamma^* e^{-i\alpha e \hat{\varphi}/\hbar} |1\rangle \langle 2|, \quad (1.22)$$

describes the hopping of the electron between sites, in the presence of a quantized electric field.

Note the following: Due to the commutation relation between $\hat{Q}$ and $\hat{\varphi}$ (Eq. 1.16), the operator $\exp(i\alpha e \hat{\varphi}/\hbar)$ shifts the charge on the node $a$ by an amount $\alpha e$. In other words, if $|q\rangle$ is an eigenstate of $\hat{Q}$ with eigenvalue $q$, then $|q + \alpha e\rangle \equiv \exp(i\alpha e \hat{\varphi}/\hbar) |q\rangle$ is an eigenstate of $\hat{Q}$ with eigenvalue $q + \alpha e$. Looking again at Eq. (1.22), we see that electron hopping is accompanied by a change in the charge on the node as we anticipated earlier.

Finally, we note that $\hat{\varphi}$ can be expressed in terms of the boson operators $b$ and $b^\dagger$ as $\hat{\varphi} = -i\sqrt{\hbar/2C\omega}(b^\dagger - b)$. Thus, $H_e$ is given by

$$H_e = \gamma e^{\lambda(b^\dagger - b)} |2\rangle \langle 1| + \gamma^* e^{-\lambda(b^\dagger - b)} |1\rangle \langle 2|, \quad (1.23)$$

with $\lambda = \alpha e/\sqrt{2\hbar\omega C}$. This makes it clear that processes occur where the electron absorbs or emits photons while hopping between sites 1 and 2.

In Chapter 3 of this thesis, we use a device similar to the two level system we have considered above to detect photons at a given energy that are produced by a voltage biased quantum point contact.

1.4 Graphene: Dirac Fermions

Graphene is a material consisting of a single atomic layer of carbon atoms. It can be thought of as a single layer of graphite. Over several decades, its unusual band structure has stimulated the interest of theorists [28, 29, 30, 31, 32]. Until recently however, it was not clear whether graphene is simply a theorist’s toy or whether it exists in nature. This question was only
answered in 2004 when the group of Andrei Geim in Manchester reported the successful fabrication of graphene devices for electron transport experiments [33]. Subsequent experimental studies, particularly those performed in the quantum Hall regime [34, 35], confirmed the theoretical prediction that low energy excitations are described by a two dimensional Dirac equation. Several tutorials [36, 37, 38, 39, 40] and reviews [41, 42, 43, 44] provide an overview of recent developments. Here we discuss only the very basics.

Graphene has a two dimensional honeycomb lattice. A honeycomb lattice consists of two triangular sublattices denoted $A$ and $B$. These are arranged such that each $A$ ($B$) sublattice site is at the centroid of the triangle formed by its nearest neighbor sites. These neighboring sites all belong to the $B$ ($A$) sublattice. This is illustrated in Fig. 1.5. The distance between nearest neighbors on the lattice is $a \simeq 1.42\text{Å}$.

The Brillouin zone corresponding to this lattice is hexagonal. It is de-
Figure 1.6. Left panel: The Brillouin zone of the graphene lattice. The $K$ and $K'$ points at the corners of the Brillouin zone are indicated by open and filled circles respectively. The three $K$ points are connected by reciprocal lattice vectors and are therefore equivalent. The same holds for the $K'$ points. There is no reciprocal lattice vector connecting $K$ and $K'$, and these two corners are inequivalent. The center of the Brillouin zone is at the point $\Gamma$. The dashed lines indicate the contour along which the dispersion is plotted in the right panel. Right panel: The energy dispersion of graphene along the lines $\Gamma K, K K', K' \Gamma$ in the Brillouin zone. At the two inequivalent corners $K$ and $K'$, the conduction and valence bands touch at energy $E = 0$, the Fermi energy of undoped graphene. These are called Dirac points. Close to the Dirac points the dispersion of both the conduction and valence bands are linear. The associated excitations (particles or holes) are described by the Dirac equation (1.25).

The band structure of the conductance and valence bands is obtained by associating one orbital with each lattice site and constructing a tight

\begin{align}
K &= \frac{2\pi}{3a} \left( \hat{x} + \frac{1}{\sqrt{3}} \hat{y} \right), \quad K' = \frac{2\pi}{3a} \left( -\hat{x} + \frac{1}{\sqrt{3}} \hat{y} \right). \quad (1.24)
\end{align}

The band structure of the conductance and valence bands is obtained by associating one orbital with each lattice site and constructing a tight
binding Hamiltonian. To good approximation only nearest neighbor hopping has to be taken into account. The nearest neighbor hopping energy is $t \approx 3\text{eV}$. The resulting energy dispersion is shown in Fig. 1.6. It is seen that the conduction and valence bands touch. Touching occurs at so-called Dirac points situated at the corners of the Brillouin zone. The energy at which the bands touch is equal to the Fermi energy of undoped graphene. Like a semi-conductor, undoped graphene therefore has a filled valence band and an empty conduction band. Unlike a semi-conductor though, there is no energy gap between the valence and conduction bands. Graphene is therefore called a semi-metal.

We now examine the dispersion relation close to one of the two inequivalent Dirac points. To be definite, let us consider the Dirac point at $K$. We define $p = \hbar(k - K)$ as the momentum associated with the wave vector $k$ and measured from a reference point $\hbar K$. In the vicinity of $\hbar K$, the dispersion relation reads $E = \pm v|p|$ where $v = 3ta/2 \approx 10^6 \text{m/s}$ is the Fermi velocity. This describes two cones touching at the Dirac point. The positive sign refers to the conduction band while the negative sign refers to the valence band. Excitations travel at a group velocity $v_g = \nabla_p E = \pm v\hat{p}$. The magnitude of the group velocity is equal to the Fermi velocity, independent of energy. Electrons in graphene behave like massless relativistic particles, traveling at the effective speed of light $v$ regardless of their energy.

For excitations close to the Fermi energy of undoped graphene, the tight binding Hamiltonian can be expanded in momentum around either of the Dirac points. We consider here states in the vicinity of the $K$ point. A continuum description results in which the electron wave function $\Psi(r)$ is defined on the whole $x$-$y$ plane such that $\exp(iK \cdot r) \times \Psi(r)$ interpolates the value of the tight-binding wave function defined on the honeycomb lattice. The continuum wave function $\Psi$ satisfies $E\Psi = H\Psi$ where $H$ is the Dirac Hamiltonian

$$H = v(p - \alpha) \cdot \sigma + \phi.$$  (1.25)

Here $\alpha$ is $e$ times the magnetic vector potential in the $x$-$y$ plane and $\phi$ is $e$ times the electric scalar potential. Both of these have to be smooth on the scale of the inter-atomic distance $a$. In position representation, the momentum is $p = -i\hbar(\partial_x, \partial_y)$. The vector $\sigma = (\sigma_x, \sigma_y)$ contains the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (1.26)$$
The Hamiltonian acts on spinors $\Psi = (\psi_A, \psi_B)$, with $\psi_A$ the amplitude to be on the $A$ sublattice and $\psi_B$ the amplitude to be on the $B$ sublattice. This spinor degree of freedom is called pseudospin to distinguish it from the real electron spin, which does not appear in the Eq. (1.25).

The excitations around the Dirac point at $K'$ are also described by the Dirac Hamiltonian of Eq. (1.25). The fact that excitations around both Dirac points are present in weakly doped graphene results in a two-fold degeneracy of eigenstates. The associated degree of freedom is called the valley index. As long as the spatial variation of $\alpha$ and $\phi$ is smooth on the scale of the inter-atomic distance $a$, and we restrict ourselves to energies $E \ll \hbar v/a$, the valleys remain uncoupled in an infinite graphene sheet. At the edges of a finite sheet however, the valleys can be coupled by the boundary [45].

The Dirac equation gives rise to unusual transport properties. This is illustrated by the following example. Consider an electrostatic potential barrier $\phi(r)$ in the region $0 < x < L$. Usually such a potential introduces scattering. A particle incident on $\phi$ from the left ($x < 0$) is reflected back with probability $R$ and transmitted to the right ($x > L$) with probability $T = 1 - R$. If the energy $E$ of the incident particle is less than the barrier height, transmission through the barrier is strongly suppressed and the reflection probability tends to unity. For energies larger than the barrier height, the probability for transmission becomes finite. However, only in the limit $E \gg \max \phi$ does the transmission probability approach unity.

Dirac excitations in graphene do not conform to this picture. Due to Klein tunneling [46, 47], transmission is no longer strongly suppressed when the barrier height exceeds the energy of the wave function. This is illustrated in Fig. 1.7.
Figure 1.7. Conventional tunneling vs. Klein tunneling. Top panel: A potential barrier $\phi(x)$ in a conventional metal or semi-conductor. The valence band (dashed curve) and the conduction band (solid curve) to the left of the barrier (I), in the barrier region (II) and to the right of the barrier (III) are indicated. The region between the bottom of the conduction band in region (II) and the top of the valence band in regions (I) and (III) is shaded. In this region there are no propagating states connecting regions (I) and (III). To get through the barrier, an electron has to tunnel through the classically forbidden region. The tunneling amplitude falls off exponentially as a function of the barrier length. As a result transport between regions (I) and (III) is strongly suppressed. Bottom panel: A potential barrier $\phi(x)$ in graphene. In each of the regions (I), (II) and (III), the valence band (dashed curve) and conduction band (solid curve) touch. As a result there are propagating states connecting the regions (I) and (III) at all energies. There is no suppression of transport at incident energies below the barrier height. In fact, as Eq. (1.28) indicates, there is perfect transmission at normal incidence.
Furthermore, at normal incidence, transmission is always perfect. To see this, we solve the Dirac equation with $\phi(x)$ that depends only on the $x$-coordinate. We focus on incident waves that propagate normal to the potential barrier. (Normal incidence means that the wave number in the $y$-direction is zero, and the wave function only depends on the $x$ coordinate.) In this case, the time-independent Dirac equation for given energy $E$ can be rewritten as

$$\partial_x \Psi(x) = \frac{i}{\hbar v} \sigma_x [E - \phi(x)] \Psi(x)$$

and the solution is found by straightforward integration. Corresponding to any energy $E$, we find a left-incident (+) and a right-incident (−) solution

$$\Psi_{\pm}(x) = \frac{e^{\pm iEx/\hbar v}}{\sqrt{2}} \left( \begin{array}{c} 1 \\ \pm 1 \end{array} \right) \times \begin{cases} 1 & x < 0 \\ \exp \left( \mp i \int_0^x dx' \phi(x')/\hbar v \right) & 0 < x < L \\ \exp \left( \mp i \int_0^L dx' \phi(x')/\hbar v \right) & x > L \end{cases}$$

Remarkably, these wave functions each contain an incident component and a transmitted component but no reflected component. At normal incidence, the transmission probability is always unity. This is particularly striking for incident energies smaller than the height of the potential barrier where one would normally expect almost perfect reflection.

The phenomenon we have just encountered is sometimes referred to as the absence of back-scattering in graphene [48]. It has some surprising consequences. Adding disorder to a graphene sample can enhance the conductivity [49, 50]. Furthermore, disorder that is smooth on the scale of the lattice constant cannot turn graphene into an insulator [51, 52, 53].

### 1.5 Bilayer graphene

In Chapter 5 of this thesis we consider bilayer graphene, i.e. two layers of carbon atoms one on top of the other. The way the two layers are stacked is illustrated in Fig. 1.8. The $A$ sublattice of the one layer is directly above the $B$ sublattice of the other. Bilayer graphene has a unit cell containing four atoms (two per carbon layer). The Brillouin zone is identical to that of monolayer graphene. Close to the $K$ and $K'$ corners of the Brillouin zone,
Figure 1.8. A graphene bilayer consists of two monolayers stacked one on top of the other. In order to be able to distinguish the two layers, open circles were used to indicate carbon atoms in the bottom layer while filled circles were used to indicate carbon atoms in the top layer. The $B$ sublattice of the top layer is directly above the $A$ sublattice of the bottom layer.

The bilayer is described by a $4 \times 4$ long wavelength Hamiltonian [54, 55, 56]

$$H = \begin{pmatrix}
\phi & v(p_x - ip_y) & t_\perp & 0 \\
v(p_x + ip_y) & \phi & 0 & v_3(p_x - ip_y) \\
t_\perp & 0 & \phi & v(p_x + ip_y) \\
0 & v_3(p_x + ip_y) & v(p_x - ip_y) & \phi
\end{pmatrix}. \quad (1.29)$$

(We only consider the case of zero magnetic field.) The upper left $2 \times 2$ block is the Dirac Hamiltonian of Eq. (1.25) and describes the electron dynamics inside one layer. The lower right $2 \times 2$ block is obtained by interchanging the $A$ and $B$ sublattices in the Dirac Hamiltonian and describes the electron dynamics in the other layer. The two layers are coupled by matrix elements $t_\perp$ and $v_3(p_x \pm ip_y)$. The interlayer coupling parameters have values $t_\perp \simeq 0.4 \text{ eV}$ and $v_3 \simeq 10^5 \text{ m/s}$. The term proportional to
\(v_3\) breaks the isotropy of the dispersion relation, introducing a triangular distortion known as “trigonal warping”. In Chapter 5 we ignore this complication and calculate the transport properties of the bilayer for \(v_3 = 0\). A complete calculation, including the effects of trigonal warping, has subsequently been published in Refs. [57, 58].

1.6 This Thesis

1.6.1 Chapter 2

In Chapter 2 we present a derivation of the Keldysh action of a general multi-channel time-dependent scatterer in the context of the Landauer-Büttiker approach. This result is then applied in two subsequent chapters.

In general the Keldysh action of a system is defined as

\[
A = \ln \mathcal{Z},
\]

where

\[
\mathcal{Z} = \text{Tr} \left[ T^+ \exp \left\{ -i \int_{t_0}^{t_1} dt \mathcal{H}^+(t) \right\} \rho_0 T^- \exp \left\{ i \int_{t_0}^{t_1} dt \mathcal{H}^-(t) \right\} \right].
\]  

(1.30)

In this expression, \(\rho_0\) is the initial density matrix of the system. This is evolved forwards and backwards in time with two different time-dependent Hamiltonians \(\mathcal{H}^\pm(t)\). For the purpose of this introduction we set \(\mathcal{H}^\pm = H + \chi^\pm(t)Q\) where \(H\) is the actual Hamiltonian of the system, \(Q\) is a system coordinate and \(\chi^\pm(t)\) are arbitrary functions of time. (Generalization to more fields, each coupling to a different system coordinate, is straightforward.) \(\mathcal{Z}\) and \(A\) are functionals of \(\chi^\pm\). The ordering symbol \(T^+\) indicates time-ordering of operators with the largest time-argument to the left, while \(T^-\) time-orders with the largest argument to the right.

Why are we interested in this object? Let us firstly mention its most direct application, namely to evaluate time-ordered correlators of the coordinate \(Q\). This is done by taking functional derivatives with respect to \(\chi^+(t)\) and \(\chi^-(t)\). We obtain

\[
\left\langle T^- \left( \prod_{j=1}^{M} Q(t_j) \right) T^+ \left( \prod_{k=1}^{N} Q(t'_k) \right) \right\rangle
\]

\[
= \prod_{j=1}^{M} \left( -i \frac{\delta}{\delta \chi^-(t_j)} \right) \prod_{k=1}^{N} \left( i \frac{\delta}{\delta \chi^+(t'_k)} \right) \mathcal{Z}[\chi] \bigg|_{\chi=0}.
\]  

(1.31)
A less obvious application is the following. Suppose the coordinate $Q$ of the system for which we know $\mathcal{A}$ (call it system $A$) is coupled to another system ($B$). Knowing $\mathcal{A}[\chi_{\pm}]$, we can then calculate the influence that system $A$ has on system $B$. (For this reason Feynman and Vernon [8] call $\mathcal{Z}$ the influence functional.) In chapters 3 and 4 for instance, we calculate how specific measuring devices respond when coupled to a coherent conductor, starting from an expression for the Keldysh action $\mathcal{A}$ of the conductor.

Previous studies of the Keldysh action focused on weakly interacting disordered electron systems [59, 60]. We consider the Keldysh action of an arbitrary coherent conductor connected to electron reservoirs. For such systems an explicit expression for $\mathcal{A}[\chi]$ was known (see for instance Chapter 3 or Ref. [61]) only in the case where the fields $\chi_{\pm}$ couple to electrons in the reservoirs rather than in the scattering region. We wanted to consider a setup where the scattering potential depends on the state of an adjacent quantum system (Chapter 4). In Chapter 2 we therefore generalized the known result for the action $\mathcal{A}$ to the situation where the fields $\chi_{\pm}$ couple to electrons inside the scattering region.

1.6.2 Chapter 3

In Chapter 3 we analyze the operation of a quantum tunneling detector coupled to a coherent conductor. Use is made of the theory developed in Chapter 2. The coherent conductor is biased with a voltage $V$. The circuit that connects the coherent conductor to the voltage biased electron reservoirs has a finite impedance. As a result, current fluctuations in the coherent conductor are converted into voltage fluctuations on top of $V$. The fluctuations are detected as photons. The detector is capable of frequency-resolved detection. We demonstrate that for frequencies larger than $eV/2\pi\hbar$, the output of the detector is determined by two-photon processes, two-interacting-electron processes and the interference of both processes. Two-photon processes occur when the conductor emits two photons each with energy $< eV$ and both are detected. The second process occurs when the electromagnetic environment mediates an interaction between electrons in the conductor. As a result, a single photon of energy $> eV$ is emitted and picked up by the detector. We show how the individual contributions of these processes can be resolved in experiments.
1.6.3 Chapter 4

In Chapter 4 we study a charge qubit with level splitting $\varepsilon$ coupled to a quantum point contact (QPC) driven by a bias voltage $V$. The charge qubit can be realized by the lowest two eigenstates of an electron trapped in double quantum dot. The qubit Hilbert space is spanned by a state $|1\rangle$ representing the electron localized in one dot, together with a state $|2\rangle$ representing the electron localized in the other dot. Because of the coupling to the qubit, the scattering matrix of the QPC depends on the state of the qubit. We define the qubit polarization as the probability to find the qubit in state $|1\rangle$. For given $V$, we calculate the qubit polarization as a function of the qubit level splitting. Use is made of the theory developed in Chapter 2. In the limit of weak coupling, the qubit polarization shows cusps at $\varepsilon = \pm eV$. We show that, for stronger couplings, a plateau occurs for $|\varepsilon| \leq \pm eV/2$. Further increase of the coupling leads to a polarization $p_2 = [1 + \exp(\beta\varepsilon)]^{-1}$ corresponding to an effective temperature $\beta^{-1} \sim eV$.

1.6.4 Chapter 5

Here we calculate the Fermi energy dependence of the (time-averaged) current and shot noise in an impurity-free carbon bilayer ($L \ll$ width $W$), and compare with known results for a monolayer [62]. We model the interlayer coupling by means of a hopping element $t_\perp = \hbar v/l_\perp$ between nearest neighbors in different layers. Here $l_\perp$ is the inter-layer hopping length estimated to be on the order of ten times the inter-atomic distance. At the Dirac point of charge neutrality, the bilayer ($l_\perp$ finite) transmits as two independent monolayers in parallel ($l_\perp$ infinite): Both current and noise are resonant at twice the monolayer value, so that their ratio (the Fano factor) has the same $1/3$ value as in a monolayer — and the same value as in a diffusive metal. The range of Fermi energies around the Dirac point within which this pseudo-diffusive result holds is smaller, however, in a bilayer than in a monolayer (by a factor $l_\perp/L$). It was subsequently shown by Moghaddam and Zareyan [58] that this conclusion holds only for lengths less than about $50$ nm ($\approx 30$ times $l_\perp$), because we ignored the effects of trigonal warping mentioned in Sec. 1.5.

1.6.5 Chapter 6

In Chapter 6 we consider a bipolar junction in a graphene nanoribbon in the high-magnetic field regime. In a bipolar junction a potential step
creates two regions, one $p$-doped and one $n$-doped. In the $p$-doped region, the Fermi energy is in the conduction band while in the $n$-doped region it is in the valence band. We consider the regime where the Hall conductance in both the $p$-doped and $n$-doped regions is $2e^2/h$. We calculate the two-terminal conductance $G$. In the absence of intervalley scattering, the result $G = (e^2/h)(1 - \cos \Phi)$ depends only on the angle $\Phi$ between the valley isospins (= Bloch vectors representing the spinor of the valley polarization) at the two opposite edges. This plateau in the conductance versus Fermi energy is insensitive to electrostatic disorder, while it is destabilized by the dispersionless edge state which may exist at a zigzag boundary. A strain-induced vector potential shifts the conductance plateau up or down by rotating the valley isospin.

1.6.6 Chapter 7

The Chalker-Coddington network model [63] (introduced originally as a model for percolation in the quantum Hall effect) is known to map onto the two-dimensional Dirac equation [64]. In Chapter 7 we show how the network model can be used to solve a scattering problem in a weakly doped graphene sheet connected to heavily doped electron reservoirs. We develop a numerical procedure to calculate the scattering matrix with the aide of the network model. For numerical purposes, the advantage of the network model over the honeycomb lattice is that it eliminates intervalley scattering from the outset. We avoid the need to include the heavily doped regions in the network model (which would be computationally expensive), by means of an analytical relation between the transfer matrix through the weakly doped region and the scattering matrix between the electron reservoirs. We test the network algorithm by calculating the conductance of an electrostatically defined quantum point contact and comparing with the tight-binding model of graphene. We further calculate the conductance of a graphene sheet in the presence of disorder in the regime where intervalley scattering is suppressed. We find an increase in conductance that is consistent with previous studies. Unlike the tight-binding model, the network model does not require smooth potentials in order to avoid intervalley scattering.
Appendix 1.A  The Keldysh technique: an example

In this appendix we illustrate the use of the Keldysh technique with the following example. A quantum dot is connected to two reservoirs by means of tunnel barriers. One reservoir is grounded and we measure all energies relative to its Fermi energy. For times $t < 0$ the other reservoir is held at zero voltage too, and the dot is neutral and in equilibrium with the reservoirs. Consequently, the dot’s chemical potential $\mu(t)$ is zero for $t < 0$. For times $t > 0$, a time-dependent voltage $V(t)$ is applied to one reservoir. We want to calculate the expectation value of the charge on the dot, as a function of time.

The dot can be modeled as a set of independent levels that are connected to reservoirs by tunneling. They are labeled by an integer $m$. The Green function $G$ is diagonal in this basis, i.e. $G_{m,n}(t, t') = \delta_{m,n}G_m(t, t')$. Note that, since $a_m(t)$ and $a_m^\dagger(t)$ anti-commute to unity at coinciding times, the Keldysh Green function contains information about the occupation probability of levels. More precisely

$$K_m(t, t) = -i [1 - 2 n_m(t)], \quad n_m(t) = \langle a_m^\dagger(t)a_m(t) \rangle,$$  \hfill (1.32)

with $n_m(t)$ the probability that level $m$ is occupied at time $t$. The total number of electrons $n(t)$ on the dot is the sum of all the occupation probabilities $n(t) = \sum_m n_m(t)$.

The Green function $G_m(t, t')$ obeys the following equations

$$i \partial_t G_m(t, t') = [\varepsilon_m + \mu(t)] G_m(t, t')$$

$$- \int d\bar{t} \Sigma(t, \bar{t})G_m(\bar{t}, t') = \delta(t - t') I_{2\times 2},$$  \hfill (1.33a)

$${} -i \partial_{t'} G_m(t, t') = [\varepsilon_m + \mu(t')] G_m(t, t')$$

$$- \int d\bar{t} G'_m(t, \bar{t}) \Sigma(\bar{t}, t') = \delta(t - t') I_{2\times 2}.$$  \hfill (1.33b)

A chemical potential $\mu(t)$ takes into account charging effects: When charge on the dot fluctuates so that it is no longer neutral, work has to be done against the electric field of the excess charge $Q(t)$ in order to add more charge to the dot. $\mu(t)$ is proportional to $Q(t)$, the proportionality constant being the capacitance $C$ of the dot:

$$\mu(t) = \frac{1}{C} Q(t).$$  \hfill (1.34)
Assuming that the dot is neutral at time $t = 0$, $Q(t) = n(t) - n(0)$. The self-energy describes the tunneling of electrons between the dot and the reservoirs. Its components are explicitly

$$
\Sigma^{(R)}(t, t') = -iE_{\text{Th}}\delta(t - t'),
$$

(1.35a)

$$
\Sigma^{(A)}(t, t') = iE_{\text{Th}}\delta(t' - t),
$$

(1.35b)

$$
\Sigma^{(K)}(t, t') = -\frac{2E_{\text{Th}}}{\pi} \frac{1}{t - t'} \left\{ \alpha e^{-i[\psi(t) - \psi(t')]} + 1 - \alpha \right\}.
$$

(1.35c)

In this equation $E_{\text{Th}}$ is the Thouless energy, or inverse lifetime of an electron on the dot. It characterizes the time an electron spends on the dot before tunneling through one of the tunnel barriers. The phase $\psi(t)$ is the integral of the reservoir voltage $\psi(t) = \int_0^t dt' V(t')$ and $\alpha \in [0, 1]$ is a parameter that measures the relative coupling to the reservoirs. The value $\alpha = 0.5$ corresponds to equally strong couplings to both reservoirs while $\alpha = 0$ corresponds to the dot completely decoupled from the voltage-biased reservoir and $\alpha = 1$ corresponds to the dot entirely decoupled from the grounded reservoir.

We now solve Eq. (1.33), starting with the retarded component, i.e. the upper-left block, which reads explicitly

$$
i\partial_t R_m(t, t') - [\varepsilon_m + \mu(t) - iE_{\text{Th}}] R_m(t, t') = \delta(t - t'),
$$

(1.36a)

$$
-i\partial_{t'} R_m(t, t') - [\varepsilon_m + \mu(t) - iE_{\text{Th}}] R_m(t, t') = \delta(t - t').
$$

(1.36b)

The retarded Green function $R_m(t, t')$ is defined as an anti-commutator of a creation and an annihilation operator. Since the creation and annihilation operators anti-commute at coinciding times, it follows that

$$
\lim_{t-t'\to 0^+} R_m(t, t') = -i.
$$

(1.37)

Furthermore, by definition, $R_m(t, t' > t) = 0$. Imposing these conditions leads to the unique solution

$$
R_m(t, t') = -i\theta(t - t') e^{-E_{\text{Th}}(t-t')} e^{-i\varepsilon_m(t-t')} e^{-i[\phi(t) - \phi(t')]},
$$

(1.38)

where $\phi(t)$ is the integral of the chemical potential $\mu$; $\phi(t) = \int_0^t dt' \mu(t')$. We see that $R_m(t, t')$ decays exponentially as a function of $t - t'$ with lifetime $E_{\text{Th}}^{-1}$. To understand why this is, note that the two terms that constitute the retarded Green function have the following interpretation. Term $\langle a_m(t)a_m^\dagger(t') \rangle$ is the amplitude that an electron placed in level $m$
at time $t'$ will still be there at time $t$. The term $\langle a_m^\dagger(t')a_m(t) \rangle$ is (the complex conjugate of) the amplitude that, if an electron is removed from level $m$ at time $t'$, that level will still be empty at time $t$. When the dot is coupled to reservoirs, these can populate and depopulate the levels, causing the amplitudes represented by $R_m(t,t')$ to decay. Similarly, one finds that the advanced Green function is given by

$$A_m(t,t') = i\theta(t' - t) e^{-E_{Th}(t' - t)} e^{-i\varepsilon_m(t-t')} e^{-i[\phi(t) - \phi(t')]}.$$  \hfill (1.39)

It remains for us to consider the Keldysh component of Eq. (1.33). We start by taking a brief look at the Keldysh component of the self-energy. It should be thought of as consisting of two contributions.

$$\Sigma^{(K)}(t,t') i = 2iE_{Th} \left[ \alpha \sigma_V(t,t') + (1 - \alpha)\sigma_0(t,t') \right], \hfill (1.40a)$$

$$\sigma_V(t,t') = \frac{i}{\pi} \frac{e^{-i[\psi(t) - \psi(t')]} }{t - t'}.$$  \hfill (1.40b)

The $\sigma_0$ contribution accounts for the coupling to the grounded reservoir. Note that it can be written as

$$\sigma_0(t,t') = \int \frac{d\varepsilon}{2\pi} e^{-i\varepsilon(t-t')} [1 - 2f(\varepsilon)], \hfill (1.41)$$

where $f_0(\varepsilon) = \theta(-\varepsilon)$ is the zero-temperature Fermi distribution of electrons in the grounded reservoir. The $\sigma_V$ contribution similarly accounts for the presence of the reservoir with fluctuating bias voltage. Suppose for instance that the bias voltage $V$ is constant. Then $\psi(t) = Vt$ and $\sigma_V$ can be written as

$$\sigma_V(t,t') = \int \frac{d\varepsilon}{2\pi} e^{-i\varepsilon(t-t')} [1 - 2f(\varepsilon - V)].$$  \hfill (1.42)

Again the zero-temperature Fermi distribution appears, this time with the Fermi energy appropriately shifted by $V$ relative to the grounded reservoir.

The fact that the reservoir distribution functions only appear in the Keldysh component of the self-energy illustrates the following general principle. The retarded and advanced functions determine the effective one-body spectrum, while Keldysh functions determine how states are populated.

We now solve for the Keldysh component of the dot Green function. The upper right block of Eq. (1.33a) reads

$$[i\partial_t - \varepsilon_m - \mu(t) + iE_{Th}] K_m(t,t') = \int d\tilde{t} \Sigma^{(K)}(t,\tilde{t}) A_m(\tilde{t},t').$$  \hfill (1.43)
We invoke Eq. (1.36) which says that $R_m(t, t')$ is a resolvent for the differential operator appearing on the left of the above equation. Additionally we impose the initial condition that the system was in equilibrium before the time-dependent voltage was switched on. This implies the solution

$$K_m(t, t') = \int dt_1 dt_2 R_m(t, t_1) \Sigma^{(K)}(t_1, t_2) A_m(t_2, t'). \quad (1.44)$$

So, we have found $K_m(t, t')$. Are we done yet? Not quite. We still have to determine the chemical potential $\mu(t)$. In order to do this, we have Eq. (1.34) that relates $\mu(t)$ to the excess charge $Q(t)$ on the dot, which in turn is related to the Keldysh Green function at coinciding times. Putting these together, we have

$$Q(t) = -\frac{i}{2} \sum_m K_m(t, t) - K_m(0, 0). \quad (1.45)$$

We use the solution (Eq. 1.44) for $K_m$, with the explicit form of $R_m$ and $A_m$ substituted from Eqs. (1.38) and (1.39) to obtain

$$Q(t) = -\frac{i}{2} \sum_m \int_{-\infty}^0 dt_1 dt_2 e^{E_{Th}(t_1+t_2)} e^{-i\varepsilon_m(t_2-t_1)}$$

$$\times \left\{ e^{[\phi(t+t_1)-\phi(t+t_2)]\Sigma^{(K)}(t+t_1, t+t_2) - \Sigma^{(K)}(t_1, t_2)} \right\}, \quad (1.46)$$

where we have explicitly used the fact that $\mu(t)$ and hence $\phi(t)$ are zero for $t < 0$.

Let us assume that the mean level spacing $\delta\varepsilon$ is much smaller than the Thouless energy. Then we can replace

$$\sum_m e^{-i\varepsilon_m(t_2-t_1)} \rightarrow \delta(t_2 - t_1) \frac{2\pi}{\delta\varepsilon}. \quad (1.47)$$

This is substituted into Eq. (1.46). The delta-function picks out the $t_2 \rightarrow t_1$ limit of the expression marked $X$. The factor $1/(t_1 - t_2)$ in $\Sigma^{(K)}$ results in time-derivatives of $\psi$ and $\phi$, so that

$$\lim_{t_2 \rightarrow t_1} X = -\frac{2iE_{Th}}{\pi} \left[ \mu(t) - \alpha V(t) \right]. \quad (1.48)$$
Using $\mu(t) = Q(t)/C$, we obtain an integral equation for $\mu(t)$ namely

$$
\mu(t) = -\frac{2E_{Th}}{C\delta\varepsilon} \int_{-\infty}^{t} dt' e^{-2E_{Th}(t-t')} \left[ \mu(t') - \alpha V(t') \right].
$$

This is converted into a differential equation by multiplying with $e^{2E_{Th}t}$ and taking a time-derivative. Finally we obtain

$$
\frac{d}{dt}\mu(t) + \gamma \mu(t) = \Gamma V(t),
$$

with $\Gamma = 2\alpha E_{Th}/C\delta\varepsilon$ and $\gamma = 2E_{Th}(1 + 1/C\delta\varepsilon)$. We solve this, and multiply by the capacitance, to obtain the charge on the dot as a function of time

$$
Q(t) = \frac{2\alpha E_{Th}}{\delta\varepsilon} \int_{0}^{t} dt' e^{-\gamma(t-t')} V(t').
$$

It is instructive to compare this result to conclusions drawn from the following intuitive argument. We suppose that the quantum system we just analyzed is roughly equivalent to the an electric circuit where a central region with capacitance $C$ is connected to leads 1 and 2, by means of resistors $R_1$ and $R_2$ respectively. Lead 2 is grounded while a time-dependent voltage $V(t)$ is applied to lead 1. The voltage of the central region is $\mu(t)$. It is related to the excess charge $Q(t)$ on the central region by $\mu(t) = Q(t)/C$. If $I_1$ is the current flowing from the lead 1 into the central region and $I_2$ is the current flowing from the central region into lead 2 then, Ohm’s law says

$$
I_1(t) = [V(t) - \mu(t)]/R_1, \quad I_2(t) = \mu(t)/R_2.
$$

Charge conservation implies that $dQ(t)/dt = I_1(t) - I_2(t)$. Putting everything together, we obtain a differential equation for $\mu(t)$:

$$
\frac{d}{dt}\mu(t) + \gamma_{cl}\mu(t) = \Gamma_{cl} V(t),
$$

where $\gamma_{cl} = (R_1^{-1} + R_2^{-1})/C$ and $\Gamma_{cl} = 1/R_1 C$.

This has the same form as the differential equation (1.50) that we obtained previously. However, if we compare the relaxation rates $\gamma$ and $\gamma_{cl}$ we note an important difference. In the limit $\delta\varepsilon C \gg 1$, the classical relaxation rate $\gamma_{cl}$ goes to zero, while $\gamma$ obtained with the Keldysh technique remains finite. There is a good reason for this. $\gamma$ is the rate at which excess charge
relaxes into the reservoirs. The limit of large capacitance \( C \) corresponds to a situation where the Coulomb repulsion between electrons on the dot is weak. Even in this limit, excess charge on the dot should relax into the leads. The reason is that, due to the dynamics of non-interacting electrons on the dot, every once in a while, an electron tunnels into a reservoir. This process does not require that the escaping electron be “pushed off the dot” by the other electrons. The classical argument ignores the dynamics of non-interacting electrons, so that the only method for charge to leak off the dot is through Coulomb repulsion. Hence the classical and quantum results can only be expected to agree in the \( \delta \varepsilon \ll C^{-1} \) limit. Beyond this limit, the quantum mechanical analysis, based on the Keldysh technique remains valid, while the classical argument breaks down.

We can relate the resistances \( R_1 \) and \( R_2 \) of the classical theory to the parameters of the quantum theory in the \( \delta \varepsilon \ll C^{-1} \) limit. One obtains

\[
R_1^{-1} = 2\alpha \frac{E_{\text{Th}}}{\delta \varepsilon} \frac{e^2}{\hbar}, \quad R_2^{-1} = 2(1 - \alpha) \frac{E_{\text{Th}}}{\delta \varepsilon} \frac{e^2}{\hbar},
\]

where \( e^2/\hbar \) reinstates the units that were dropped in the microscopic analysis. The quantity \( E_{\text{Th}}/\delta \varepsilon \) is known to characterize the conductance of a coherent conductor, and is called the dimensionless Thouless conductance.
Bibliography


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