Appendix B

Complete-Basis Functions

Here is a brief summary of the fundamental mathematical concepts behind the Complete-Basis-Functions Parameterization, as presented in Section 9.2.2. This part is mainly based on Abramowitz [173] and Kaplan [174]. Let \( f(x) \) be given in the interval \( a \leq x \leq b \), and let

\[
\xi_1(x), \xi_2(x), \ldots, \xi_k(x), \ldots
\]  
(B.1)

be functions which are all piecewise continuous in this interval.

The set \( \{ \xi_k(x) \}_{k=1}^{\infty} \) is called complete if it can span any piecewise continuous function \( f(x) \), e.g.,

\[
f(x) = \sum_{k=1}^{\infty} c_k \xi_k(x),
\]  
(B.2)

where the coefficients \( c_k \) are given by:

\[
c_k = \frac{1}{B_k} \int_a^b f(x) \xi_k(x) \, dx, \quad B_k = \int_a^b [\xi_k(x)]^2 \, dx
\]  
(B.3)

The convergence is guaranteed by the so-called completeness theorem. Explicitly, the series

\[
R_m = \int_a^b \left( f(x) - \sum_{k=1}^{m} c_k \xi_k(x) \right) \, dx = \int_a^b (f(x) - S_m(x))^2 \, dx
\]  
(B.4)

converges to zero for sufficiently large \( m \):

\[
\lim_{m \to \infty} R_m = 0,
\]  
(B.5)

where we denoted the sequence of partial sums as \( S_m(x) \):

\[
S_m(x) = \sum_{k=1}^{m} c_k \xi_k(x)
\]  
(B.6)

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By definition, the convergence of the series of functions is equivalent to the convergence of \( S_m(x) \) to \( f(x) \):

\[
\lim_{m \to \infty} S_m(x) = f(x)  \tag{B.7}
\]

**The Fourier (Trigonometric) Series**

A **trigonometric series** is an expansion of a periodic function in terms of a sum of *sines* and *cosines*, making use of the orthogonality property of the harmonic functions. Without loss of generality, let us consider from now on the interval \([0, L]\). Let \( f(x) \) be a single-valued function defined on that interval, then its **trigonometric series** or **trigonometric expansion** is given by:

\[
\tilde{f}(x) = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k \cos \left( \frac{2\pi k}{L} \cdot x \right) + \sum_{k=1}^{\infty} b_k \sin \left( \frac{2\pi k}{L} \cdot x \right)  \tag{B.8}
\]

If the coefficients \( a_k \) and \( b_k \) satisfy certain conditions, then the series is called a **Fourier series**.

If \( f(x) \) is periodic with period \( L \), and has continuous first and second derivatives for all \( x \) in the interval, it is guaranteed that the trigonometric series of \( f(x) \) will converge uniformly to \( f(x) \) for all \( x \); This is referred to as satisfying the **Dirichlet conditions**. We shall refer in this study to the **trigonometric** series as the **Fourier** series.

**Other Sets of Functions**

If one is indeed interested in periodic functions, there is no natural alternative but using the trigonometric series. However, if one is concerned with other representations of a general function over a given interval, a great variety of other sets of functions is available, e.g.:

- **Legendre polynomials**, \( P_k(x) \):

  \[
P_k(x) = \frac{(2k-1)(2k-3)\cdots 1}{k!} \left\{ x^k - \frac{k(k-1)}{2(k-1)}x^{k-2} + \frac{k(k-1)(k-2)(k-3)}{2\cdot 4(2k-1)(2k-3)}x^{k-4} - \cdots \right\}  \tag{B.9}
\]

  which can also be defined via Rodrigues’ formula:

  \[
P_0(x) = 1 \quad P_k(x) = \frac{1}{2^k k!} \frac{d^k}{dx^k} (x^2 - 1)^k, \quad k = 1, 2, \ldots  \tag{B.10}
\]

If \( f(x) \) satisfies the **Dirichlet conditions** mentioned earlier, then there will exist a Legendre series expansion for it in the interval \(-1 < x < 1\). For illustration, the first 10 **Legendre polynomials** are plotted in Figure B.1.
• **Bessel Function of the First Kind and of Order** $l$, $J_l(x)$:

$$J_l(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{l+2k}}{2^{l+2k} \cdot k! \cdot \Gamma(l + k + 1)}$$  \hfill (B.11)

with $\Gamma(\alpha)$ as defined in Eq. 1.36. Given a fixed $l \geq 0$, the functions \( \{\sqrt{x}J_l(\lambda_k x)\}_{k=1}^{\infty} \) form an orthogonal complete system over the interval $0 \leq x \leq 1$.

• **Hermite polynomials**, $H_k(x)$:

$$H_k(x) = (-1)^k \exp \left\{ x^2 \right\} \frac{d^k}{dx^k} \left( \exp \left\{ -x^2 \right\} \right), \; k = 0, 1, \ldots$$  \hfill (B.12)

The Hermite polynomials form a complete set of functions over the infinite interval $-\infty < x < \infty$, with respect to the weight function $\exp \left( -\frac{1}{2} x^2 \right)$.

• **Chebyshev polynomials of the First Kind**, $T_k(x)$:

$$T_k(x) = \frac{k}{2} \sum_{r=0}^{[k/2]} \frac{(-1)^r}{k-r} \binom{k-r}{r} (2x)^{k-2r}, \; k = 0, 1, \ldots$$  \hfill (B.13)

The Chebyshev polynomials of the First Kind form a complete set of functions over the interval $[-1, 1]$ with respect to the weight function $\frac{1}{\sqrt{1-x^2}}$. 

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**Figure B.1:** The First 10 *Legendre* Polynomials.
Higher Dimensions

An expansion by means of a complete set of functions can be generalized for higher dimensions. For illustration, let us consider the two-dimensional case of the trigonometric series. The functions $\cos(\frac{2\pi k}{L} \cdot x) \cdot \cos(\frac{2\pi l}{L} \cdot y)$, $\sin(\frac{2\pi k}{L} \cdot x) \cdot \cos(\frac{2\pi l}{L} \cdot y)$, $\cos(\frac{2\pi k}{L} \cdot x) \cdot \sin(\frac{2\pi l}{L} \cdot y)$, and $\sin(\frac{2\pi k}{L} \cdot x) \cdot \sin(\frac{2\pi l}{L} \cdot y)$ form an orthonormal complete system of functions in the box $[(0, 0), (0, L), (L, 0), (L, L)]$. Given a function in that domain, $f(x, y)$, its expansion can then be written in the form:

$$f(x, y) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \lambda_{kl} \left\{ a_{kl} \cos(\frac{2\pi k}{L} x) \cos(\frac{2\pi l}{L} y) + b_{kl} \sin(\frac{2\pi k}{L} x) \cos(\frac{2\pi l}{L} y) + c_{kl} \cos(\frac{2\pi k}{L} x) \sin(\frac{2\pi l}{L} y) + d_{kl} \sin(\frac{2\pi k}{L} x) \sin(\frac{2\pi l}{L} y) \right\}$$

(B.14)

Corollary

An infinite series of complete basis functions converges to any “reasonably well behaving” function. Hence, it is straightforward to approximate a given function with a finite series of those functions, i.e., by cutting its tail from a certain point. In principle, the sum $S_m(x)$ (Eq. B.6) can always be found to a desired degree of accuracy by adding up enough terms of the series. For practical applications, the corollary is that every function can be approximated using a series of complete basis functions, to whatever desired or practical accuracy. Moreover, this corollary can be easily generalized to any desired dimension.