Frequency dependence of the photonic noise spectrum in an absorbing or amplifying diffusive medium

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Abstract. A theory is presented for the frequency dependence of the power spectrum of photon current fluctuations originating from a disordered medium. Both the cases of an absorbing medium ("grey body") and of an amplifying medium ("random laser") are considered in a waveguide geometry. The semiclassical approach (based on a Boltzmann-Langevin equation) is shown to be in complete agreement with a fully quantum mechanical theory, provided that the effects of wave localization can be neglected. The width of the peak in the power spectrum around zero frequency is much smaller than the inverse coherence time, characteristic for black-body radiation. Simple expressions for the shape of this peak are obtained, in the absorbing case, for waveguide lengths large compared to the absorption length, and in the amplifying case, close to the laser threshold.

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1 Introduction

The noise power spectrum of a black body is frequency independent for frequencies below the absorption band width. The inverse of the band width is the coherence time \( \tau_{\text{coh}} \) of the radiation [1], which for a black body is the longest relevant time scale — hence the white noise spectrum \( P(\Omega) \) for \( \Omega < 1/\tau_{\text{coh}} \). In a weakly absorbing, strongly scattering medium there appear two longer time scales: the absorption time \( \tau_a \) and the time \( L^2/D \) it takes to diffuse (with diffusion constant \( D \)) through the medium (of length \( L \)). As a consequence, \( P(\Omega) \) for such a weakly-absorbing medium (sometimes called a "grey body") starts to decay at much lower frequencies than for a black body having the same coherence time.

Although there is by now a substantial literature on the theory of grey-body radiation [2–7], the results have been limited to either the zero or high-frequency limits of the noise spectrum (or, equivalently, to short or long photodetection times). In the present work we remove this limitation, by computing \( P(\Omega) \) for a diffusive medium for arbitrary ratios of \( \Omega, 1/\tau_a \), and \( D/L^2 \). We compare two different approaches in a waveguide geometry: one which is fully quantum mechanical (based on random-matrix theory [7,8]) and another which is semiclassical (based on a Boltzmann-Langevin equation [9]). Each method has its advantages and disadvantages: the quantum theory includes interference effects, which are ignored in the semiclassical theory, but it is mathematically more involved. Complete agreement between the two approaches is obtained in the limit that the waveguide length \( L \) is much smaller than the localization length (equal to the mean free path times the number of propagating modes).

The results for absorbing media can be applied directly to linear amplifiers, by formally changing the sign of the temperature and the absorption time. Loudon and coworkers [10,11] used this relationship to calculate the noise power spectrum of a waveguide without disorder. The generalization to a diffusive medium presented here describes a random laser [12] below threshold.

The outline of this paper is as follows. We start with the semiclassical approach, presenting a general solution of the Boltzmann-Langevin equation in Section 2 and applying it to a waveguide geometry in Section 3. The quantum mechanical approach is developed in Section 4. For the quantum theory we need the correlator of reflection and transmission matrices at different frequencies. These are calculated in the Appendix, using the randommatrix method of reference [13]. We discuss our findings in Section 5.

2 Semiclassical theory

Starting point of the semiclassical theory is the Boltzmann-Langevin equation for photons of reference [9]...
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Fig. 1. Thermal radiation (solid arrow) is incident through port $S_0$ on an absorbing disordered medium (shaded). The outgoing radiation (dashed arrows) is absorbed by photodetectors.

We first consider an absorbing medium (in equilibrium at temperature $T$), leaving the amplifying case for the end of this section. We make the diffusion approximation, valid if the mean free path $l$ is the shortest length scale in the system (but still large compared to the wavelength). The fluctuating number density $n(\omega, r, t)$ and current density $j(\omega, r, t)$ of photons at frequency $\omega$, position $r$, and time $t$ are related by [9]

$$ j = -D \frac{\partial n}{\partial t} + \mathbf{L}_1, $$

$$ \frac{\partial n}{\partial t} + \nabla \cdot j = D \xi_\alpha^{-2} (\rho f - n) + \mathbf{L}_0. $$

Here $D = cl/3$ is the diffusion constant, $\xi_\alpha = \frac{D}{\tau_\alpha}$ is the absorption length (with $\tau_\alpha$ the absorption time), $\rho = 4\pi \omega^2 (2\pi c)^{-3}$ is the density of states (not counting polarizations), and $f = \exp[\ln(\omega/kT) - 1]^{-1}$ is the Bose-Einstein function. We assume $\xi_\alpha \gg l$. The fluctuating source terms $\mathbf{L}_0$ and $\mathbf{L}_1$ have zero mean and correlators

$$ \mathbf{L}_0(\omega, r, t) = \delta(\omega - \omega') \delta(t - t') \delta(r - r') $$

$$ \times D \xi_\alpha^{-2} (2f \bar{n} + \rho f + \bar{n}), $$

$$ \mathbf{L}_1(\omega, r, t) = 2\delta_{\alpha \beta} \delta(\omega - \omega') \delta(t - t') \delta(r - r') $$

$$ \times Dn(1 + \bar{n}/\rho). $$

The cross-correlator of $\mathbf{L}_0$ and $\mathbf{L}_1$ is given in reference [9], but will not be needed. Combining equations (2.1, 2.2) we find equations for the mean $\bar{n}$ and the fluctuations $\delta n$ of the photon number density $n = \bar{n} + \delta n$.

$$ \frac{1}{D} \frac{\partial \bar{n}}{\partial t} + \frac{\partial^2 \bar{n}}{\partial r^2} = -\frac{\rho f}{\xi_\alpha^2}, $$

$$ \frac{1}{D} \frac{\partial \delta n}{\partial t} + \frac{\partial^2 \delta n}{\partial r^2} = -\frac{\partial}{\partial r} \cdot \mathbf{L}_1 - \frac{\mathbf{L}_0}{D}. $$

We present a general solution for the multiport geometry of Figure 1. Thermal radiation is incident through the port $S_0$ and can leave the system via ports $S_0, S_1, S_2, \ldots$, where it is absorbed by photodetectors. The corresponding boundary conditions are $n(\omega, r, t)|_{r \in S_p} = n(\omega, t)\delta_{p0}$. We assume that the closed boundaries $\Sigma$ of the system (with volume $V$) are perfectly reflecting. The separation of the ports is of order $L \gg l$. In what follows we assume detection of outgoing radiation in a narrow frequency interval $\delta\omega$ around $\omega$. We require that $\delta\omega$ is small both compared to $\omega$ and to $1/\tau_{\text{coh}}$. To minimize the notations in this section we omit the frequency argument $\omega$ and use units in which $\delta\omega \equiv 1$. (We will reinsert $\delta\omega$ in the next section.)

The Green function of the differential equations (2.4, 2.5) in the Fourier representation with respect to the time argument satisfies

$$ \left( \frac{\partial^2}{\partial r^2} - \xi_\alpha^{-2} + \frac{i\Omega}{D} \right) G(r, r', \Omega) = \delta(r - r'). $$

(Fourier transforms are defined as $f(\Omega) = \int_{-\infty}^{\infty} dt e^{i\Omega t} f(t)$.) For frequency resolved detection we require $\Omega \ll \delta\omega$. We impose the boundary conditions

$$ G(r, r', \Omega)|_{r \in S_p} = 0, \quad p = 0, 1, 2, \ldots, $$

$$ \Sigma : \frac{\partial G(r, r', \Omega)}{\partial \mathbf{r}}|_{r \in \Sigma} = 0, $$

where $\Sigma$ denotes the outward normal direction to the surface $\Sigma$. We consider separately the mean and the fluctuations of the photon number and current densities.

2.1 Mean solution

The average photon density satisfying equation (2.4) can be expressed in Fourier representation in terms of the Green function (2.6),

$$ \bar{n}(r, \Omega) = -2\pi \rho f \xi_\alpha^{-2} \delta(\Omega) \int d\Omega' G(r, r', 0) $$

$$ + \bar{n}_{\text{in}}(\Omega) \int dS' \cdot \frac{\partial G(r, r', \Omega)}{\partial r'}. $$

Substituting this formula into the expression for the current (2.1) and integrating over the area $S_p$ one obtains the mean outgoing current $I_p$ through port $p \neq 0$.

$$ I_p(\Omega) = 2\pi \rho D f \xi_\alpha^{-2} \delta(\Omega) \int dS \cdot \int d\Omega' G(r, r', 0) $$

$$ - D \bar{n}_{\text{in}}(\Omega) \int dS_\alpha \int dS_\beta \frac{\partial^2 G(r, r', \Omega)}{\partial r_\alpha \partial r_\beta}. $$

(Summation over the repeating Greek indices is implied.) The first term $\propto \delta(\Omega)$ is the time-independent mean thermal radiation from the medium. The second term is that part of the mean radiation entering through port 0 that leaves the medium through one of the other ports. (The restriction to $p \neq 0$ is not essential but simplifies the general formulas considerably, so we will make this restriction in what follows.)
2.2 Fluctuations

The fluctuations in the number density follow in a similar way from the Green function and equation (2.5),

\[ \delta n(r, \Omega) = \frac{1}{D} \int d r' \frac{\partial G(r', r', \Omega)}{\partial r} \frac{\partial G(r', r', \Omega)}{\partial r} \]

\[ + \delta n_m(\Omega) \int d S' \cdot \frac{\partial G(r, r', \Omega)}{\partial r}. \]  

(2.10)

The fluctuation of the current density is then given by equation (2.1),

\[ \delta J_{\alpha}(r, \Omega) = \int d r' \left( G_{\alpha \beta}(r, r', \Omega) L_1(r', \Omega) + \frac{\partial G(r, r', \Omega)}{\partial r} L_0(r', \Omega) \right) \]

\[ - D \delta n_m(\Omega) \int d S' \frac{G_{\alpha \beta}(r, r', \Omega)}{S_0}. \]  

(2.11)

We have defined

\[ G_{\alpha \beta}(r, r', \Omega) = \frac{\partial G(r, r', \Omega)}{\partial r} + \delta_{\alpha \beta}(r - r'). \]  

(2.12)

We seek the correlator of the current fluctuations

\[ C_{\alpha \beta}(r, \Omega; r', \Omega') = \delta J_{\alpha}(r, \Omega) \delta J_{\beta}(r', \Omega') \]  

(2.13)

for \( r \in S_p, r' \in S_q \) with \( p, q \neq 0 \). With the help of equations (2.3, 2.11) it can be expressed as

\[ \text{see equation (2.14) above.} \]

Following reference [9], we have neglected the term \( \propto \delta n_m \) in equation (2.11) (smaller by a factor \( l/L \)) and the cross-correlator \( \delta G_0(\Omega) \) (smaller by a factor \( l/\xi_0 \)).

We now integrate \( r \) and \( r' \) over \( S_p \) and \( S_q \) to obtain the correlator of the total currents through ports \( p \) and \( q \),

\[ C_{pq}(\Omega, \Omega') = \int d S_p \int d S_q \frac{G_{\alpha \beta}(r, \Omega; r', \Omega')}{S_p S_q} \]

\[ = C_{pq}^{(1)}(\Omega, \Omega') + C_{pq}^{(2)}(\Omega, \Omega'). \]  

(2.15)

The first term \( C_{pq}^{(1)} \) contains the contribution from the terms linear in the number density \( \bar{n} \) in equation (2.14).

Performing integration by parts and using equations (2.6–2.8) we find that this term vanishes for \( p \neq q \). For \( p = q \) it contains the mean current,

\[ C_{pq}^{(1)}(\Omega, \Omega') = \delta_{pq} \bar{I}_p(\Omega + \Omega'). \]  

(2.16)

For a time-independent mean current \( \bar{I}_p \) one has a white-noise spectrum \( C_{pq}^{(1)}(\Omega, \Omega') = 2\pi \delta_{pq} \delta(\Omega + \Omega') \bar{I}_p \). This is the usual shot noise, corresponding to Poissonian statistics of the current fluctuations. The second term \( C_{pq}^{(2)} \) describes the deviations from Poissonian statistics. It arises from terms in equation (2.14) that are quadratic in \( \bar{n} \).

Performing again an integration by parts, one finds

\[ C_{pq}^{(2)}(\Omega, \Omega') = \frac{2D}{\rho} \int d S_p \int d S_q \frac{\partial G(r, r', \Omega)}{\partial r} \frac{\partial G(r', r', \Omega)}{\partial r} \]

\[ \times \left( \frac{\partial \bar{n}(r, \Omega)}{\partial r} - \frac{\partial \bar{n}(r, \Omega)}{\partial r} \right) \frac{\partial G(r, r', \Omega)}{\partial r} \frac{\partial G(r', r', \Omega)}{\partial r}. \]  

(2.17)

Equation (2.17) together with equation (2.8) is the result that we need for our analysis of the frequency dependence of the noise spectrum.

2.3 Amplifying medium

The extension of our general formulas to an amplifying medium (in the linear regime below the laser threshold) is straightforward [9]: we assume that the frequency \( \omega \) at which we are detecting the radiation is close to the frequency of an atomic transition with \( \omega \) (on average) \( \overline{N_{\text{upper}}} \) and \( \overline{N_{\text{lower}}} \) atoms in the upper and lower state, so that the Bose-Einstein function can be replaced by the population inversion factor \( \gamma = \frac{\overline{N_{\text{upper}}} - \overline{N_{\text{lower}}}}{\overline{N_{\text{lower}}}} \). This factor is negative in the amplifying case (when \( \overline{N_{\text{upper}}} > \overline{N_{\text{lower}}} \) ), with \( \gamma = -1 \) for a complete population inversion. (Equivalently, one can evaluate \( \gamma \) at a negative temperature [11], with \( T \to 0^+ \) for complete inversion.) An amplifying medium has a negative absorption time \( \tau_a = \frac{\xi_0}{D} \).

We can account for this by taking \( \xi_0 \) imaginary. With these two substitutions for \( \gamma \) and \( \xi_0 \) our formulas for an absorbing medium carry over to the amplifying case.

3 Waveguide geometry

For the application of our general formulas we consider a waveguide geometry (see Fig. 2). The waveguide has
length $L$ and cross-sectional area $A$, corresponding to $N = \omega^2 A/4\pi c^2$ propagating modes (not counting polarizations) at frequency $\omega$. We abbreviate $s = L/\xi_a$. We consider a stationary incident current $I_0 = cA\delta\omega\bar{n}_m/4 = (N\delta\omega/2\pi p)\bar{n}_m$, and calculate the noise power spectrum of the transmitted current,

$$P(\Omega) = \int_{-\infty}^{\infty} dt e^{i\Omega t} \delta I(t)\delta I(0)$$

In terms of the correlator of the previous section, one has

$$C_{11}(\Omega, \Omega') = 2\pi P(\Omega)\delta(\Omega + \Omega')$$

### 3.1 Absorbing medium

We calculate the noise power from equations (28, 217), using the Green function

$$G(x, x', \Omega) = -\frac{\xi_a}{\sinh \left[\left(\frac{x_<}{\xi_a}\right)\sqrt{1 - i\Omega \tau_a}\right] \sinh \left[\left(\frac{s - x_>}{\xi_a}\right)\sqrt{1 - i\Omega \tau_a}\right]}$$

$$\times \frac{\sinh \left[\left(\frac{1}{\xi_a}\right)\sqrt{1 - i\Omega \tau_a}\right] \sinh \left[\left(\frac{s}{\xi_a}\right)\sqrt{1 - i\Omega \tau_a}\right]}{\sinh \left[\left(\frac{1}{\xi_a}\right)\sqrt{1 - i\Omega \tau_a}\right]}$$

(32)

where $x_<$ and $x_>$ are the smallest and largest of $x, x'$ respectively. The mean photon density is time independent. In Fourier representation one has, from equation (28),

$$n(x, \Omega) = 2\pi \delta(\Omega) \frac{\rho f}{\sinh s}$$

$$\times \left[ \sinh s - \sinh (x/\xi_a) - \sinh (s - x/\xi_a) \right]$$

$$+ 2\pi \delta(\Omega) \bar{n}_m \frac{\sinh (s - x/\xi_a)}{\sinh s}$$

(33)

The mean current $I = I_{th} + I_{trans}$ is the sum of the thermal radiation from the medium

$$I_{th} = \frac{4DF}{c\xi_a}(N\delta\omega/2\pi)\tanh (s/2)$$

(34)

and the transmitted incident current

$$I_{trans} = \frac{4DF_0}{c\xi_a \sinh s}$$

(35)

Substitution of equations (32, 33) into equation (217) yields the super-Poissonian noise $P - I$ as a sum of three terms, $P - I = P_{th} + P_{trans} + P_{ex}$, with

$$P_{th}(\Omega) = \frac{8DF^2}{c\xi_a} \left(N\delta\omega/2\pi\right)$$

$$\times \int_0^s ds' \frac{\cosh (s - s') - \cosh s'}{\sinh s} K(s', s)$$

(36)

$$P_{trans}(\Omega) = \frac{8DF_0^2}{c\xi_a} \left(2\pi/N\delta\omega\right)$$

$$\times \int_0^s ds' \frac{\cosh^2 (s - s')}{\sinh^2 s} K(s', s)$$

(37)

$$P_{ex}(\Omega) = \frac{16DF_0}{c\xi_a}$$

$$\times \int_0^s ds' \frac{\cosh s' - \cosh (s - s')}{\sinh s} \cosh (s - s') K(s', s)$$

(38)

We have defined

$$K(s', s) = \frac{\sinh (s'\sqrt{1 - i\Omega \tau_a})}{\sinh (s\sqrt{1 - i\Omega \tau_a})}$$

(39)

The two terms $P_{trans}$ and $P_{th}$ describe separately the noise power of the transmitted incident current and of the thermal current from the medium. The term $P_{ex}$ is the excess noise due to the beating of the incident radiation with the thermal fluctuations from the medium.

The three contributions are plotted separately in Figure 3. For $L \gg \xi_a$ the frequency dependence simplifies to

$$P_{th}(\Omega) = \frac{I_{th}}{1 + \zeta}$$

(310)

$$P_{trans}(\Omega) = \frac{c\xi_a I_{trans}}{16D} \left(2\pi/N\delta\omega\right)$$

$$\times \left(1 - e^{-2\pi(\zeta - 1)} + \frac{3\zeta + 2}{\zeta^2 + \zeta} \right)$$

(311)

$$P_{ex}(\Omega) = \frac{I_{trans}}{\zeta + \xi^2}$$

(312)

where we have defined

$$\zeta = \text{Re} \sqrt{1 - i\Omega \tau_a} = \left[\frac{1}{2}(1 + \Omega^2 \tau_a^2)^{1/2} + \frac{1}{2}\right]^{1/2}$$

(313)

As discussed in reference [9] (for the zero-frequency case) the result for $P_{trans}$ requires that the incident radiation is in a thermal state, at some temperature $T_0$ (The quantity $f(\omega, T_0) = I_0(2\pi/N\delta\omega)$ is the corresponding value of the Bose-Einstein function.) There is no such requirement for $P_{th}$ and $P_{ex}$, which are independent of the incident state. For $T_0 \gg T$ we may generally neglect $P_{th}$ and $P_{ex}$ relative to $P_{trans}$, so that $P = I_{trans} + P_{trans}$. However, if the incident radiation is in a coherent state, then $P_{trans} \equiv 0$ and since for sufficiently large $I_0$ we may neglect $P_{th}$, we have in this case $P = I_{trans} + P_{ex}$. The contribution $P_{th}$ is important mainly in the absence of external illumination, when $P = I_{th} + P_{th}$.
3.2 Amplifying medium

The results for an amplifying medium are obtained by the substitution $\xi_a \rightarrow i \xi_a$, $f \rightarrow N_{\text{uppc}}(N_{\text{lowpc}} - N_{\text{uppc}})^{-1}$, cf. Section 2.3. The frequency dependence of $P_{\text{th}}, P_{\text{trans}}$, and $P_{\text{ex}}$ following from equations (3.6–3.8) is plotted in Figure 4 for lengths $L$ below the laser threshold at $L = \pi \xi_a$.

3.3 Cross-correlator

In the absence of any incident radiation, the noise $P = \tilde{I}_{\text{th}} + P_{\text{th}}$ is due entirely to the thermal fluctuations in the medium. The current fluctuations at the two ends of the waveguide are correlated, as measured by the cross-correlator

$$P_{12}(\Omega) = \int_{-\infty}^{\infty} dt \, e^{i\Omega t} \delta I_1(t) \delta I_2(0)$$

(3.14)

4 Comparison with quantum theory

A fully quantum mechanical theory for the photocount distribution of a disordered medium was developed in references [7,8]. In this section we verify that it agrees with the semiclassical results of the previous section. We consider the same system of Figure 2, a disordered waveguide with a photodetector at one end and a stationary current incident at the other end. We assume that the incident current originates from a thermal source at temperature $T_0$. The photocount distribution is the distribution...
Fig. 5. Frequency dependence of the cross-correlator of the outgoing current at the two ends of the waveguide, in the absence of any external illumination. Computed from equation (3.15) for the absorbing case (lower panel) and amplifying case (upper panel).

of the number of photons $n(t)$ counted (with unit quantum efficiency) in the time interval $(0,t)$. Substitution of $I = dn/dt$ in the definition (3.1) of the noise power $P(\Omega)$ leads to a relation with the variance $\text{Var} n(t)$ of the photocount,

$$P(\Omega) = -\Omega^2 \int_0^\infty dt \text{Var} n(t) \cos \Omega t,$$  \hspace{1cm} (4.1a)

$$\text{Var} n(t) = -\frac{2}{\pi} \int_0^\infty d\Omega \Omega^{-2} P(\Omega) (\cos \Omega t - 1)$$  \hspace{1cm} (4.1b)

The variance can be separated into two terms, $\text{Var} n(t) = \bar{n}(t) + \kappa(t) = I + \kappa(t)$, with $\kappa(t)$ the second factorial cumulant. The term $I$, substituted into equation (4.1a), gives the frequency-independent shot noise contribution $I$ to the power spectrum,

$$P(\Omega) = I - \Omega^2 \int_0^\infty dt \kappa(t) \cos \Omega t$$  \hspace{1cm} (4.2)

The cumulant $\kappa = \kappa_{\text{trans}} + \kappa_{\text{th}} + \kappa_{\text{ex}}$ contains separate contributions from the transmitted incident radiation and thermal fluctuations in the medium, plus an excess contribution from the beating of the two. These contributions have an exact representation in terms of the $N \times N$ reflection and transmission matrices $r(\omega)$, $t(\omega)$ of the waveguide [7, 8],

$$\kappa_{\text{trans}}(t) = \int_0^\infty \int_0^\infty \frac{d\omega'}{2\pi} \int_0^\infty \frac{d\omega''}{2\pi} L(\omega - \omega', t)$$  \hspace{1cm} (4.3)

$$\times f(\omega, T_0)f(\omega', T_0)\text{Th}(\omega)T(\omega'),$$

$$\kappa_{\text{th}}(t) = \int_0^\infty \int_0^\infty \frac{d\omega'}{2\pi} \int_0^\infty \frac{d\omega''}{2\pi} L(\omega - \omega', t)$$  \hspace{1cm} (4.4)

$$\times f(\omega, T)f(\omega', T)\text{Th}(\omega)Q(\omega)'Q(\omega),$$

$$\kappa_{\text{ex}}(t) = \int_0^\infty \int_0^\infty \frac{d\omega'}{2\pi} \int_0^\infty \frac{d\omega''}{2\pi} L(\omega - \omega', t)$$  \hspace{1cm} (4.5)

$$\times 2f(\omega, T_0)f(\omega', T)\text{Th}(\omega)Q(\omega)'Q(\omega),$$

where we have defined

$$L(\omega, t) = \int_0^t dt' \int_0^t dt'' \exp[i(\omega t'' - \omega t')] = 2\omega^{-2}(1 - \cos \omega t),$$  \hspace{1cm} (4.6)

$$Q(\omega) = 1 - r(\omega)r(\omega),$$  \hspace{1cm} (4.7)

$$T(\omega) = t(\omega)t(\omega).$$  \hspace{1cm} (4.8)

Substitution into equation (4.2) gives the corresponding contributions to the noise power $P = P_{\text{trans}} + P_{\text{th}} + P_{\text{ex}},$

$$P_{\text{trans}}(\Omega) = \frac{1}{2} \int_0^\infty \frac{d\omega'}{2\pi} f(\omega, T_0)f(\omega + \Omega, T_0)$$  \hspace{1cm} (4.9)

$$\times \text{Th}(\omega)T(\omega + \Omega) + \{\Omega \rightarrow -\Omega\},$$

$$P_{\text{th}}(\Omega) = \frac{1}{2} \int_0^\infty \frac{d\omega'}{2\pi} f(\omega, T)f(\omega + \Omega, T)$$  \hspace{1cm} (4.10)

$$\times \text{Th}(\omega)Q(\omega)'Q(\omega + \Omega) + \{\Omega \rightarrow -\Omega\},$$

$$P_{\text{ex}}(\Omega) = \frac{1}{2} \int_0^\infty \frac{d\omega'}{2\pi} 2f(\omega, T_0)f(\omega + \Omega, T)$$  \hspace{1cm} (4.11)

$$\times \text{Th}(\omega)Q(\omega + \Omega) + \{\Omega \rightarrow -\Omega\}$$

As in the previous section, we assume a frequency-resolved measurement in an interval $\delta \omega \ll \omega, 1/\tau_{\text{coh}}$. With $\Omega \ll \delta \omega$. We may then omit the integral over $\omega$ and approximate the argument $\omega \pm \Omega$ in the functions $f$ by $\omega$. We take the ensemble average $\langle \ldots \rangle$ of the noise power, in which case the contributions from $\pm \Omega$ are the same. Finally, we insert the incident current $f_0 = f(\omega, T_0)N\delta \omega/2\pi$, to arrive at

$$P_{\text{trans}}(\Omega) = (2\pi/N\delta \omega)_0^2 \langle N^{-1} \text{Th}(\omega)T(\omega + \Omega) \rangle,$$  \hspace{1cm} (4.12)

$$P_{\text{th}}(\Omega) = (N\delta \omega/2\pi)^2 f^2(\omega, T)\langle N^{-1} \text{Th}(\omega)Q(\omega + \Omega) \rangle,$$  \hspace{1cm} (4.13)

$$P_{\text{ex}}(\Omega) = 2f(\omega, T)\langle N^{-1} \text{Th}(\omega)Q(\omega + \Omega) \rangle$$  \hspace{1cm} (4.14)
It remains to evaluate the ensemble averages. This is done in the Appendix, by extending the approach of reference [13] to correlators of reflection and transmission matrices at different frequencies. The calculation applies to the diffusive regime that the length $L$ of the waveguide is large compared to the mean free path $l$, but still small compared to the localization length $N l$. (The absorption length $\xi \alpha$ is also assumed to be $\gg l$.) The results are

$$\langle N^{-1} \text{Tr} T(\omega) T(\omega + \Omega) \rangle = \frac{8D}{c \xi \alpha} \int_0^s ds' K(s', s) \times \frac{\cosh^2(s - s')}{\sinh^2 s},$$  

(4.15)

$$\langle N^{-1} \text{Tr} Q(\omega) Q(\omega + \Omega) \rangle = \frac{8D}{c \xi \alpha} \int_0^s ds' K(s', s) \times \frac{[\cosh s' - \cosh(s - s')]^2}{\sinh^2 s},$$  

(4.16)

$$\langle N^{-1} \text{Tr} T(\omega) Q(\omega + \Omega) \rangle = \frac{8D}{c \xi \alpha} \int_0^s ds' K(s', s) \times \frac{\cosh(s - s') \cosh s' - \cosh^2(s - s')}{\sinh^2 s},$$  

(4.17)

where $s = L/\xi \alpha$ and the kernel $K(s', s)$ is defined in equation (A.29). The combination of equations (4.12-4.17) agrees precisely with the results (3.6-3.8) of the semiclassical theory. The quantum theory is more general than the semiclassical theory, because it can describe the effects of wave localization. The method of reference [13] gives corrections to the above results in a power series in $L/N l$. We will not pursue this investigation here.

### 5 Discussion

We have presented a theory for the frequency dependence of the noise power spectrum $P(\Omega)$ in an absorbing or amplifying disordered waveguide. The frequency dependence is governed by two time scales, the absorption or amplification time $\tau_a$ and the diffusion time $L^2/D$, both of which are assumed to be much greater than the coherence time $\tau_{coh}$. A simplified description is obtained, in the absorbing case, for lengths $L$ much greater than the absorption length $\xi \alpha = \sqrt{D/\tau_a}$, and, in the amplifying case, close to the laser threshold at $L = \pi \xi \alpha$. We will discuss these two cases separately.

#### 5.1 Absorbing medium

The general formulas (3.6-3.8) for $P = \bar{I} + P_{th} + P_{trans} + P_{ex}$ simplify for $L \gg \xi \alpha$ to equations (3.10-3.12). To characterize the frequency dependence we define the characteristic frequency $\Omega_c$ as the frequency at which the super-Poissonian noise has dropped by a factor of two:

$$P(\Omega_c) - \bar{I} = \frac{1}{2} \left( P(0) - \bar{I} \right).$$  

(5.1)

In the absence of any external illumination ($I_0 = 0$) we have, from equation (3.10),

$$P = \bar{I}_{th} \left( 1 + \frac{f}{1 + \zeta} \right), \quad \bar{I}_{th} = \frac{4Df}{c \xi \alpha} (N \delta \omega / 2 \pi).$$  

(5.2)

with $\zeta = \Re \sqrt{1 - i \Omega \tau_a}$, hence $\Omega_c = 17/\tau_a$. If the illumination is in the coherent state from a laser, then we have, from equation (3.12),

$$P = \bar{I}_{trans} \left( 1 + f \frac{1 + 2 \zeta}{\zeta + \zeta^2} \right), \quad \bar{I}_{trans} = \frac{8D I_0}{c \xi \alpha} e^{-s},$$  

(5.3)

here $\Omega_c = 9/\tau_a$. In both these cases the diffusion time does not enter in the frequency dependence. This is different for illumination by a thermal source at temperature $T_0$ much greater than the temperature of the medium. From equation (3.11), with $f_0 = f(\omega, T_0)$, we then have

$$P_{trans} (\Omega) = \bar{I}_{trans} \times \left( 1 + \frac{f_0}{2} e^{-s} \left[ \frac{1 - e^{-2s (-1)} - 3 \zeta + 2}{\zeta - 1} \right] \right).$$  

(5.4)

The characteristic frequency $\Omega_c = (64D/L^2 \tau_a^2)^{1/4}$ now contains both the diffusion time and the absorption time.

#### 5.2 Amplifying medium

In the amplifying case the noise power becomes more and more strongly peaked near zero frequency with increasing amplification. Close to the laser threshold at $s = \pi$ the frequency dependence of $P_{th}$ for small frequencies $\Omega \tau_a \ll 1$ has the form

$$P_{th} = \frac{Z \bar{I}_{th}^2}{2 \pi (\Omega^2 \tau_a^2 + 4(1 - s / \pi)^2)},$$  

$$\bar{I}_{th} = \frac{4f}{Z (\pi - s)}. \quad (5.5)$$

Here again $Z = (c \xi \alpha / 2D) (2 \pi / N \delta \omega)$. Close to threshold the peak in the noise power spectrum has a Lorentzian
to the reflection matrix from the right by
$$SL$$
We$$r' = -r$$
Finally, we note the fundamental difference between
the time scales appearing in the noise spectrum for photons,
on the one hand, and electrons, on the other hand.
The absorption or amplification time $$\tau_\alpha$$ obviously has
no electronic analogue. The diffusion time $$L^2/D$$ appears
in both contexts, however, the electronic noise spectrum
remains frequency independent for $$\Omega > D/L^2$$ [9].

The reason for the difference is screening of electronic charge. As a
result, the characteristic frequency scale for electronic current
fluctuations is the inverse scattering time $$D/l'$$, which
is much greater than the inverse diffusion time $$D/L^2$$.

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Appendix A: Correlators of reflection
and transmission matrices

To compute the noise power spectrum in the quantum
mechanical approach of Section 5, we need the correlators
of reflection and transmission matrices $$t(\omega_\pm)$$ and $$r(\omega_\pm)$$
at two different frequencies $$\omega_\pm = \omega \pm \Omega/2$$ (For $$\Omega \ll \omega$$ this
is the same as the correlator at frequencies $$\omega$$ and $$\omega + \Omega$$.)
We calculate these correlators for a waveguide geometry
in the diffusive regime, by extending the equal-frequency
($$\Omega = 0$$) theory of Brouwer [15].

Upon attachment of a short segment of length $$\delta L$$ to
one end of the waveguide of length $$L$$, the transmission
and reflection matrices change according to
$$t \rightarrow t_{\delta L}(1 + rt_{\delta L})t$$,
$$r \rightarrow r_{\delta L} + t_{\delta L}(1 + rt_{\delta L})rt_{\delta L}^{-1}$$,
where the superscript T indicates the transpose of a
matrix. (Because of reciprocity the transmission matrix from
left to right equals the transpose of the transmission matrix
from right to left.) The transmission matrix $$t_{\delta L}$$ of
the short segment at frequency $$\omega_\pm$$ may be chosen pro-
tional to the unit matrix,
$$t_{\delta L} = \left(1 - \frac{\delta L}{2l'} - \frac{\delta L}{2c'\tau_\alpha} \pm \frac{i\delta L}{2c'} \frac{\Omega}{2c'} \right)I$$

The mean free path $$l' = 4l/3$$ and the velocity $$c' = c/2$$
represent a weighted average over the $$N$$ transverse modes
in the waveguide.

Unitarity of the scattering matrix dictates that the re-
flexion matrix from the left of the short segment is related
to the reflection matrix from the right by $$r_{\delta L} = -r_{\delta L}^*$$. We
abbreviate $$r_{\delta L} \equiv \delta r$$. The matrix $$\delta r$$ is symmetric (because
of reciprocity), with zero mean and variance
$$(\delta r_{kl} \delta r_{mn}^*) = (N + 1)^{-1}(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm})dL/l'$$

The resulting change in the matrix products $$tt^\dagger$$ and $$rr^\dagger$$ is
$$tt^\dagger \rightarrow (1 - \delta L/l' - \delta L/c'\tau_\alpha)tt^\dagger + (r\delta rt)(\delta rt)^\dagger$$
$$+ r\delta rt^\dagger + (r\delta rt^\dagger)^\dagger$$,
$$rr^\dagger \rightarrow (1 - 2\delta L/l' - 2\delta L/c'\tau_\alpha)rr^\dagger + (r\delta rr)(\delta rr)^\dagger + \delta r^\dagger \delta r$$
$$+ r\delta rr^\dagger + (r\delta rr^\dagger)^\dagger - r\delta r - (r\delta r)^\dagger$$

The frequency $$\Omega$$ does not appear explicitly in these in-
crements.

We define the following ensemble averages
$$\mathcal{R} = \langle tN^{-1}Tr(I - rr^\dagger) \rangle$$,
$$\mathcal{C} = \langle tN^{-1}Tr(I - r_- r_+^\dagger) \rangle$$,
$$\mathcal{T} = \langle tN^{-1}Tr(tt^\dagger) \rangle$$,
where $$r, t$$ are evaluated at frequency $$\omega$$ and $$r_\pm, t_\pm$$ at fre-
quency $$\omega \pm \Omega/2$$. Similarly, we define the correlators

$$C_{rr} = \langle tN^{-1}Tr((I - r_- r_+^\dagger)I - r_+ r_-^\dagger) \rangle$$,
$$C_{rt} = \langle tN^{-1}Tr((I - r_- r_+^\dagger)t_+ t_-^\dagger) \rangle$$,
$$C_{tt} = \langle tN^{-1}Tr(t_+ t_-^\dagger t_+ t_-^\dagger) \rangle$$.

We will see that, in the diffusive regime, these 6 quantities
satisfy a coupled set of ordinary differential equations in $$L$$. The
diffusive regime corresponds to the large-$$N$$ limit, in
which the length $$L$$ of the waveguide is much less than the
localization length $$Nl$$. In this limit we may replace equation
(3) by $$(\delta r_{kl} \delta r_{mn}^*) = (\delta L/Nl')\delta_{km}\delta_{ln}$$. In the
large-$$N$$ limit we may also replace averages of products
of traces by products of averages of traces. From equation
(4) we thus obtain the differential equations
$$t' \frac{dL}{dL} = 2\gamma(1 - \mathcal{R}) - \mathcal{R}^2$$,
$$t' \frac{dC}{dL} = 2\gamma(1 + i\Omega\tau_\alpha)(1 - C) - C^2$$,
$$t' \frac{dT}{dL} = -\gamma T - RT$$,
$$t' \frac{dC_{rr}}{dL} = -(4\gamma + C + C^* + 2\mathcal{R})C_{rr} + 2\mathcal{R}(\mathcal{R} + 2\gamma)$$,
$$t' \frac{dC_{rt}}{dL} = -(3\gamma + C + C^* + \mathcal{R})C_{rt} - TC_{rr} + 2(\mathcal{R} + \gamma)T$$,
$$t' \frac{dC_{tt}}{dL} = -(2\gamma + C + C^*)C_{tt} - 2TC_{rt} + 2T^2$$,
with the definition $$\gamma = t'/c'\tau_\alpha$$. The initial conditions are
that each of these 6 quantities goes to 1 for $$L \rightarrow 0$$.

This set of differential equations may be simplified fur-
fther if we assume, as we did in the semiclassical theory,
that the mean free path is small compared to both the absorption length and the length of the waveguide. All 6 quantities (A 5–A 10) are of order $\sqrt{\gamma}$, which is $\ll 1$ if $l' \ll c'\tau_0$, so that we obtain in leading order:

\[ l' \frac{dR}{dL} = 2\gamma - R^2, \quad (A 17) \]
\[ l' \frac{dC}{dL} = 2\gamma(1 + 1\Omega\tau_a) - C^2, \quad (A 18) \]
\[ l' \frac{dT}{dL} = -RT, \quad (A 19) \]
\[ l' \frac{dC_{rr}}{dL} = -(C + C^* + 2R)C_{rr} + 2R^2, \quad (A 20) \]
\[ l' \frac{dC_{rt}}{dL} = -(C + C^* + R)C_{rt} - TC_{rr} + 2RT, \quad (A 21) \]
\[ l' \frac{dC_{tt}}{dL} = -(C + C^*)C_{tt} - 2TC_{rt} + 2T^2 \quad (A 22) \]

As initial condition we should now take that the product of each quantity with $L$ remains finite when $L \to 0$. Although the differential equations are coupled, they may be solved separately for $R, C, T, C_{rr}, C_{rt}, C_{tt}$, in that order. In terms of the rescaled length $s = (2\gamma)^{1/2}L/l' = L/\xi_0$, the results are:

\[ R = \frac{(2\gamma)^{1/2}}{\tanh s}, \quad (A 23) \]
\[ C = \frac{(2\gamma)^{1/2}\sqrt{1 + 1\Omega\tau_a}}{\tanh s\sqrt{1 + 1\Omega\tau_a}}, \quad (A 24) \]
\[ T = \frac{(2\gamma)^{1/2}}{\sinh s}, \quad (A 25) \]
\[ C_{rr} = \frac{(8\gamma)^{1/2}}{\sinh^2 s} \int_0^s ds' K(s', s) \cosh^2 s', \quad (A 26) \]
\[ C_{rt} = \frac{(8\gamma)^{1/2}}{\sinh^2 s} \int_0^s ds' K(s', s) \cosh(s - s') \cosh s', \quad (A 27) \]
\[ C_{tt} = \frac{(8\gamma)^{1/2}}{\sinh^2 s} \int_0^s ds' K(s', s) \cosh^2(s - s'), \quad (A 28) \]

where the kernel $K$ is defined by:

\[ K(s', s) = \left| \frac{\sinh s'\sqrt{1 + 1\Omega\tau_a}}{\sinh s\sqrt{1 + 1\Omega\tau_a}} \right|^2 \quad (A 29) \]

These are the expressions used in Section 4 (where we have also substituted $\sqrt{2\gamma} = 4D/c^2\xi_0$). The remaining integrals over $s'$ may be done analytically, but the resulting expressions are rather lengthy so we do not record them here. For $\Omega = 0$ our results reduce to those of Brouwer [13] (up to a misprint in Eq (13c) of that paper, where the plus and minus signs in the expression between brackets should be interchanged).

References