"Mind Your P's and Q's!"
On the Proper Interpretation of Modal Logic*
Göran Sundholm

1.

Of old logic is concerned with judgements. The traditional "Aristotelian" form of judgement was

(1) S is P,

that is, subject/copula/predicate form, where the subject S and predicate P are suitable "terms". It clearly allows for applications of modality in two places.\(^1\)

On the one hand, one can ascribe necessity to the whole judgement (1):

(3) The judgement that \(S \text{ is } P\) is necessary,

and, on the other hand, one can modalise the predicate-term P:

(4) \(S \text{ is } \text{nec-}P\).\(^2\)

The form (3) has been called *de dicto*; here the modality appears to be imposed upon a *dictum*, namely that what is being said or stated in the judgement (1). The form (4), on the other hand, has been called *de re*; in it a modal property, namely \(\text{nec-}P\), is ascribed to an object.\(^3\)

2.

In the nineteenth century the form of judgement considered in logic was changed through the work of Bolzano [1837] and Frege [1879]. The relevant

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* Text of a lecture delivered at LOGICA 2002. It constitutes a first attempt to extend the semantical approach advocated by Per Martin-Löf also to modal notions, and I owe him thanks, not only for my preferred semantical framework, but also for many helpful conversations. The material was also presented at a Paris workshop on Phenomenology and Logic, March 2001, and at an Oskar Becker Tagung, Hagen, February 2002. I am indebted to organisers and participants alike.

1 For reasons of space, I here confine myself largely to the sole modality of necessity; nevertheless, the treatment is readily extendible also to other modalities, in particular to possibility.

2 The natural language rendering \(S \text{ is necessarily } P\) seems ambiguous between (3) and (4): after all, an adverb is capable of modifying either verbs or an adjective. If the necessity pertains to S's being P we have alternative (3), and if it pertains to P we have alternative (4).

3 Kneale [1962] is the *locus classicus* concerning the traditional conception.
form of judgement is no longer the bipartite (1). In its place one uses unary form in which truth is ascribed to a suitable (abstract) entity, be it a proposition(-in-itself) or a Fregean Thought:

\[(5) \text{ Proposition } A \text{ is true.}\]

On this reading the customary logical connectives are what Frege called *Gedankenueflege.* Thus, semantically they are functions taking (one or more) proposition(s) into a proposition. The negation sign ‘\(\neg\)’, for instance, stands for a function that takes a proposition as an argument and yields a proposition as value. Around 1930 the metamathematical revolution in logic changed all this: the expressions of the formal languages of logic are stripped of their content and are turned into mere formulae, that is, strings in certain freely generated algebras over suitable alphabets. The WFF’s now do double duty as the formalistic simulacrums of both propositions and asserted theorems. The early, path-breaking work in technical modal logic, in particular, the designing of the syntax, and the ensuing search for a suitable formal semantics, took place after the metamathematical turn around 1930. Therefore, strictly speaking, contentual aspects are absent in current modal logic, when it is pursued in a meta-theoretical fashion. I will however, ignore this circumstance and treat of the modal formalism as if it had content.

3.

Inspection of the form of judgement (5) reveals three places amenable to modalisation. First one can modalise the proposition A:

\[(6) \text{ proposition } \Box A \text{ is true.}\]

This, then, is the form that is implicit in the syntax of current modal logic, when its systems, in deliberate contravention of their design, are read with

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4 Bolzano adapted the Aristotelian form of judgements into a form of propositions: his Platonist propositions-in-themselves have the form \(V \text{ has } W\), where \(V\) and \(W\) are “ideas-in-themselves”. Frege, on the other hand, was more radical and used a mathematical “function-applied-to-argument” form \(P(a)\). The details of these developments in logical theory concerning the proper form of an (atomic) proposition \(A\) are dealt with at length in my [forthcoming], and need not detain us overly here. It should be stressed, though, that a proposition \(A\) is not a linguistic entity but an abstract object that can be referred to or otherwise expressed using linguistic expressions.

5 Sundholm[1997] offers a reasonably self-contained exposition, following Per Martin-Löf [1983] and other works cited there, of my preferred treatment of the notions of proposition, proof of a proposition, and truth, judgement(candidate), demonstration and correctness, as well as judgement made and knowledge.

6 See my LOGICA 2001 lecture, i.e. Sundholm [2002].

7 The early work of MacColl is an exception that springs to mind; nevertheless, Lewis and Langford [1932] remains the pivot around which the introduction of modal systems revolves.
content. Secondly, one can modalise the truth that is ascribed to the proposition A:

(7) proposition A is a necessary truth (is Nec-true).

In such a judgement, it appears, one does not ascribe mere truth to the proposition A, but rather something stronger, namely necessary truth. It remains to be determined what relation, if any, that holds between the forms (6) and (7). Leaving that task aside temporarily, one can finally modalise the judgement (5) as a whole:

(8) The judgement
Proposition A is true
is necessary.⁹

These observations, for sure, are neither very deep nor surprising, but it is striking to note that necessity constrains a different “bearer” of modality in each of the three cases. In (6) it is applied to a proposition, in (7) to a form of judgement, namely [...] is true], where the dots hold open a place for propositions, while in (8) necessity is claimed for a whole judgement (assertion). In the first instance, the necessity box □ of current modal logic appears to cater only for alternative (6).¹⁰

4.

Our immediate task, then, is to investigate relations and reductions among alternatives (6) - (8). Since the work of Kripke (Kanger, Hintikka, Montague, ...) in the 1950’s, the syntax in (6) is, of course, intimately tied to a matching semantics cast in terms of “possible worlds”. Recall that a possible worlds model M is a set-theoretical object

⁸ Some, among whom Enrico Martino and Gabriele Usberti [1991], would point to standard formalisations of first-order predicate logic, where one and the same WFF A may serve both as building material for other WFF’s, as well as a derivable theorem, and, with this as reason, question the legitimacy, and/or utility, of the distinction between the proposition A and the judgement [proposition A is true]. To my mind, their denial is question-begging, but a full treatment of the issue would require a separate paper. Here I confine myself to the observation that the need for the distinction is borne out both by historical precedents (Frege, Russell, Heyting, [early]Carnap, ...), as well as independent systematic arguments primarily drawn from speech-act theory.

⁹ Again the natural language rendering Proposition A is necessarily true is ambiguous. If the necessity pertains to truth we have alternative (7) and if it pertains to A’s being true we have alternative (8).

¹⁰ There is no confusion between use and mention here. The formal systems of modal logic are metalogical; so quotation marks are rightly left out. Please recall that □ is a metamathematical object about which one speaks employing the name ‘□’. Thus, in the main text, I refer to □ by using its name ‘□’, but not by using ‘□’ that, of course, refers to the name ‘□’.
where (i) \( K = < W, R > \) is a frame, and (ii) \( V \) is a valuation function over that frame. Here (iii) \( W \) is a non-empty set of "possible worlds" and (iv) \( R \) is a two-place "accessibility" relation over \( W \), that is, \( R \subseteq W \times W \), where for each "world" \( \alpha \in W \), \( V_{\alpha} = V(\alpha) \) is a function from the natural numbers \( \mathbb{N} \) into the set of Boolean "truth-values" \( \{T, F\} \). This function \( V_{\alpha} \) is used to give truth-values to the propositional letters:

\[
(9) \quad M \text{ satisfies } p, \text{ iff } V_{\alpha}(p) = T.
\]

This "semantics", of course, does not give a meaning to the WFF's of modal logic, but defines a meta-theoretic property that some of the WFF's may have and others not, just in the same way that some natural numbers are divisible by 37 and others not. Numbers do not, however, acquire mysterious semantic powers through the definition of what it is to be divisible by 37, nor do the WFF's via the definition of "truth" in a model; they still remain strings in the freely generated algebra whether they happen to fall under a certain inductively defined the concept, or not, as the case may be.

5.

If, instead of the above formalistic simulacrum, we avail ourselves of a contentual possible-worlds idiom, necessity can be explained as truth in, or with respect to, all possible worlds. In such a case, however, the contentual notion of necessity that is thus catered for is not \( \Box \), that is, the propositional connective from (6), but the strengthening of ordinary simple truth for propositions into the necessary truth that is employed in (7):

\[
(10) \quad \text{proposition A is a necessary truth } =_{df} A \text{ is true in all possible worlds.}
\]

Thus the possible-worlds idiom yields a novel form of judgement by constraining the notion of truth that is ascribed to a proposition in a judgement of the usual form (5).

I now consider the relationship between the respective modalities in (6) and (7). In other words, how does the truth of the proposition \( \Box A \) relate to the necessary truth of the proposition \( A \)? Consider the (anodyne) propositional connective \( T \) explained by the truth-condition

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11 Property speaking, what is defined by the standard recursion is, not a property of WFF's, but a two-place relation \( \text{SAT}(M, \phi) \) between modal models \( M \) and WFF's \( \phi \).

12 Section 4 applies considerations from Sundholm [2002] to the case of modal "semantics".

13 Recall that the views of Saul Kripke, the foremost formal semanticist of \( \Box \) and \( \Diamond \), differs from those of Saul Kripke, the metaphysician of Naming and Necessity, who voices doubts about the philosophical usefulness of the set-theoretical apparatus [1972/1980, p. 345/48, fn. 15].
(11) \( T(A) \) is true if and only if \( A \) is true.\(^{14}\)

This \( T \)-connective constitutes a propositional internalisation of the form of judgement (7), in that the judgement

(12) proposition \( T(A) \) is true

and the judgement (5) are equi-assertible.\(^{15}\) The relation between judgements (6) and (7) is now the same as between (12) and (5); the propositional connective \( \Box \) effectuates an internalisation of the same kind as the \( T \)-connective does, but not of the form of judgement involving mere truth, but of the stronger form of judgement that ascribes necessary truth to a proposition. One important difference between the forms of judgement [... is true] and [...] Nec.-true] and their matching propositional internalisations \( T \) and \( \Box \) is that the latter allow iteration and embedding under connectives.\(^{16}\)

Thus, for instance

\[
T(T(T(A) \& \neg T(A \vee T(A))))
\]

and

\[
\Box(\Box A \supset \neg \Box A \& \neg (\Box \Box A \& \Box A))
\]

are both propositions when \( A \) is a proposition. The forms of judgement given by (5) and (7) (or (10) ), on the other hand, cannot be so iterated. In each of these forms of judgement, the open place can be taken only by a proposition, but not by a judgement: for instance, when \( A \) is a proposition,

\[ A \text{ is true is true] would, nevertheless, be a category mistake, as well as lacking in grammar.}\]

\(^{14}\) Strictly speaking, in order to explain a proposition, it is not enough merely to give its truth-condition along the lines of (11), (5), and (6) combined with (10). I prefer, following Martin-Löf, to cast the semantics of propositions in terms of canonical proof-objects; see Sundholm [1997, §3], where also the notion of a truth-condition is explained in terms of proof-objects:

\[ A \text{ is true} = \text{Proof}(A) \text{ exists.} \]

The canonical proof-condition for the \( T \)-connective is

(11') \( t(a) \) is a canonical proof object for the proposition \( T(A) \), when \( a \) is proof-object for \( A \).

\(^{15}\) Thus, the judgements (5) and (12) are equi-assertible, but do not have the same assertion-condition, since the propositions \( A \) and \( T(A) \) have different canonical proofs.

\(^{16}\) Note here that \( T \)-connective is not the Tarskian truth-predicate \( Tr^r \). The latter is not a connective, but a propositional function, with the inductively defined set of \( \text{wff} \)'s a its range of definition:

\[ Tr^r(x) : \text{prop, provided that } x : \text{wff}. \]

The \( T \)-connective, on the other hand, when applied to a proposition (rather than a metamathematical object) yields a proposition and not an element of the set of \( \text{wff} \)'s.
The form of judgement (7), as well as its propositional internalisation (6), are both concerned with an alethic modality that operates on propositions by constraining their truth into necessary truth or, alternatively, by yielding another modal proposition. The necessity in (8) is of another kind: here it is a judgement rather than a proposition that is qualified. Up till now I have been concerned mainly with necessity in relation to propositions, and the contrasting notion of judgement has accordingly been used in a fairly loose sense, with little or no attention paid to a number of subtle distinctions. However, when the relevant bearer of modality is an entire judgement rather than a proposition, more care is called for.

Propositions are explained in terms of truth-conditions. These truth conditions are explained in terms of (canonical) proof-objects. A judgement J, on the other hand, is explained in terms of the knowledge that is required for having the right to make it.\footnote{Following Dummett [1973, p. 362], I see judgement as the interiorization of the exterior linguistic act of assertion, contra Frege [1918, p. 62], who took assertion to be the exteriorization of the interior act of judgement. At the linguistic level, the meaning of a declarative S is explained in terms of its assertion-condition, and the declarative expresses an assertion- or judgement-candidate.} Judgement/assertion-candidates have assertion-conditions, whereas propositions have truth-conditions.\footnote{In mathematics, the declarative sentence \footnote{17 Martin-Löf [1983, p. 26]. It is well-known from elementary logic texts that the legitimacy of the question “Is the sentence S true?” provides a criterion for whether S is a declarative or not. Martin-Löf’s elucidation of judgement suggests a matching question-criterion for assertoric force. An utterance of a declarative S is assertoric if it is wrong not to (be able to) answer an interlocutor’s question: “How do you know?” An assertion made through an assertoric utterance of a declarative contains an illocutionary knowledge-claim. See Sundholm [1988, p. 17] and many later places, e.g., [1999, pp. 120-123].} 

<table>
<thead>
<tr>
<th>MODALITY</th>
<th>p true</th>
<th>p false</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) p true</td>
<td>p true</td>
<td>p true</td>
</tr>
<tr>
<td>(2) p false</td>
<td>p false</td>
<td>p false</td>
</tr>
<tr>
<td>(3) p impossible</td>
<td>p necessarily false</td>
<td>p necessarily false</td>
</tr>
<tr>
<td>(4) p not impossible</td>
<td>p not necessarily false</td>
<td>p not necessarily false</td>
</tr>
<tr>
<td>(5) p impossibly false</td>
<td>p necessary true</td>
<td>p necessary true</td>
</tr>
<tr>
<td>(6) p not impossibly false</td>
<td>p not necessary true</td>
<td>p not necessary true</td>
</tr>
</tbody>
</table>

expresses a judgement-candidate in the relevant “proposition is true” form,
namely:

(15) proposition Prime(17) is true,

where Prime (17) is a proposition explained as a set of appropriate proof-objects. A proposition in its turn is true when a certain truth-maker exists.

In other words, the judgement-candidate (15) is explained in terms of the judgement-candidate

(16) Proof(Prime(17)) exists.\(^{20}\)

The assertion-condition for this judgement-candidate is that an element of the set Proof(Prime(17)) be found (or known); ultimately thus, an act of judgement/assertion by means of an assertoric utterance of the declarative (14) is permitted only when a judgement-candidate of the fully explicit form

(17) \(c\) is a proof-(object) of the proposition Prime(A)

is already known.

Clearly, judgement(-candidate), when thus elucidated, is an \textit{epistemic} notion. The object of an act of judgement, that is, the judgement made, is a piece of knowledge and takes the form that a judgement-candidate is known. It is expressed by employing assertoric force to the assertion-candidate expressed by the matching declarative. At the level of judgement(-candidate)s the appropriate epistemic notion of necessity is, at least in mathematics, that of an \textit{apodictic} judgement, that is, the necessity that pertains to a judgement-candidate that is known (demonstrated, grounded): what is known to be so, cannot be otherwise.\(^{21}\)

7.

One of the earliest modern treatments of modality was that offered by Oskar Becker [1930]. His book \textit{Mathematische Existenz} [1927] was instrumental at

\(^{18}\) Previously, e.g. [1999, p. 122, fn. 5], I used the (overburdened) term \textit{statement} for what a declarative expresses. See also Van der Schaar [2001]. \(^{19}\) Here I differ from other anti-realist positions, for instance that of Michael Dummett, where only classical propositions have truth-conditions, whereas constructive propositions have assertion-conditions. I hold that propositions have truth-conditions also constructively. Assertion-conditions, on the other hand, pertain to judgement or (better) \textit{assertion} candidates, but not to propositions, be they classical or constructivist. \(^{20}\) The notion of existence used here is not that of the (propositional) existential quantifier, be it classical or intuitionistic, but the \textit{constructive} notion of existence employed by Brouwer, and perhaps first explicitly formulated by Hermann Weyl. (See Sundholm [1994].) When \(\alpha\) is a type ("general concept"), \(\alpha \exists\) is a judgement candidate, with the assertion condition: In order to have the right to make the judgement \(\alpha \exists\) one must already know some judgement \(c : \alpha\).

Becker formulates six modalities cast in two different styles. Reformulating Becker’s table yields a more illuminating display of the alternatives in which an interesting structure is revealed:

\[
\begin{array}{ccc}
\text{p is} & \text{necessary} & \text{true} & \text{possible} \\
\text{impossible} & \text{false} & \text{non-necessary}
\end{array}
\]

Becker then notes a tripartite “Law of Excluded Third”:

1. \( p \) is either true or false (not true);
2. \( p \) is either necessary (not impossible) or impossible;
3. \( p \) is either possible (not impossible) or impossible.

If we spell this out using epistemic necessity (“knownness”), and use a constructive semantics, Becker’s observations makes also constructive sense. First note that \( p \), the bearer of Becker’s six modalities, which, after the fashion of early modal logic, has at least the syntax of a proposition, will now have to be a judgement(-candidate) in either of the two forms

**A true** and **A false**, where \( A \) is a proposition. These judgement(-candidates) are then centred around the three central, positive modalities **true**, **necessary**, and **possible** from the (reformulated) Becker schema, which is recast as:

**A true** is \[ \text{necessary \ true \ possible} \]

**A false** is \[ \text{necessary \ true \ possible} \]

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21 I am indebted to my colleague Maria van der Schaar for reminding me of Kant’s treatment in the Jäsche Logik § 30; see also her contribution to the present volume.
22 Heyting [1931, p.107 (Eng. tr. p.53)]; see also Troelstra [1990].
24 [1930, §2, pp. 15-16].
25 When \( A \) is a proposition **A is false** is a judgement (-candidate) that may be asserted when one has obtained a function \( f \) from \( \text{Proof}(A) \) to the empty set \( \emptyset \). In other words, the assertion-condition is that, for a suitable function \( f \), one must know a judgement \( f(x) : \text{Proof}(\bot) \), given that \( x : \text{Proof}(A) \). Note here that this function \( f \) is not a proof-object for \( \neg A \). Nevertheless, since \( \neg A =_{df} A \vdash \bot \), a proof-object for \( \neg A \) is readily given in terms of the function \( f \), namely \( \vdash (A, \bot, (x)f) : \text{Proof}(\neg A) \). (Here \( (x)f \) is the (“lambda”-) abstraction of \( f \), with respect to the variable \( x \).
Applying the epistemic translation of Becker’s three central modalities necessary, true, and possible, one obtains the following six alternatives:

\[
\begin{align*}
\text{A true necessary} & \rightarrow \text{A true known;} \\
\text{A true true} & \rightarrow \text{A true knowable;}^{26} \\
\text{A true possible} & \rightarrow \text{A false not known;} \\
\text{A false necessary} & \rightarrow \text{A false known;} \\
\text{A false true} & \rightarrow \text{A false knowable;} \\
\text{A false possible} & \rightarrow \text{A true not known.}
\end{align*}
\]

Recasting these alternatives in a more compact horizontal display allows three constructive versions of the Law of Excluded Middle to emerge:

\[
\begin{array}{ccc}
\text{I} & \text{II} & \text{III} \\
\text{A true known} & \text{A true knowable} & \text{A false known} \\
\text{A true not known} & \text{A true not knowable} & \text{A false not known}
\end{array}
\]

For each of the three bipartite alternatives, it is ruled out that both components should fail to hold. For instance, assume that \text{A false} fails to be known. Then the matching claim below the line holds trivially, since obviously \text{A false} is not known. An application of Martin-Löf [1995, pp. 194-195] where it is shown that when \text{A true} is not knowable, the judgement candidate \text{A false} is knowable, allows for an emendation of a Becker schema in a less trivial final form:

\[
\begin{align*}
\text{CONSTRUCTIVE LAWS OF EXCLUDED MIDDLE} \\
\text{A true known} & \rightarrow \text{A true knowable} & \rightarrow \text{A false known} \\
\text{A true not known} & \rightarrow \text{A false knowable} & \rightarrow \text{A false not known}
\end{align*}
\]

When a claim above the line fails to hold, the matching claim below the line does hold:

\[
\text{TERTIUM NON DATUR!}
\]

References:


\[^{26}\text{The combination} \quad \text{A true true} \]

is neither pleonastic nor nonsensical; the boldface occurrence pertains to propositions, whereas the second is truth (“correctness”) for judgements, which is explained as knowability (demonstrability).


