

PROOF THEORY AND MEANING*

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The meaning of a sentence determines how the truth of the proposition expressed by the sentence may be proved and hence one would expect proof theory to be influenced by meaning-theoretical considerations. In the present Chapter we consider a proposal that also reverses the above priorities and determines meaning in terms of proof. The proposal originates in the criticism that Michael Dummett has voiced against a realist, truth-theoretical, conception of meaning and has been developed largely by him and Dag Prawitz, whose normalization procedures in technical proof theory constitute the main technical basis of the proposal.

In a subject not more than 20 years old, and where much work is currently being done, any survey is bound to be out of date when it appears. Accordingly I have attempted not to give a large amount of technicalities, but rather to present the basic underlying themes and guide the reader to the ever-growing literature. Thus the Chapter starts with a general introduction to meaning-theoretical issues and proceeds with a fairly detailed presentation of Dummett's argument against a realist, truth-conditional, meaning theory. The main part of the Chapter is devoted to a consideration of the alternative proposal using 'proof-conditions', instead of truth-conditions, as the key concept. Finally, the Chapter concludes with an introduction to the type theory of Martin-Löf.

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1. THEORIES OF MEANING, MEANING THEORIES AND TRUTH THEORIES

A *theory of meaning* gives, one might not unreasonably expect, a general account of, or view on, the very concept of meaning: what it is and how it functions. Such theories *about* meaning, however, do not hold undisputed rights to the *appellation*; in current philosophy of language one frequently encounters discussions of theories of meaning for particular languages. Their task is to specify the meaning of all the sentences of the language in question. Following Peacocke [1981] I shall use the term 'meaning theory' for the latter, language-relative, sort of theory and reserve 'theory of meaning' for the former. Terminological confusion is, fortunately, not the only connection between meaning theories and theories of meaning. On the contrary, the main reason for the study and attempted construction of meaning theories is that one hopes to find a correct theory of meaning through reflection on the various desiderata and constraints that have to be imposed on a satisfactory meaning theory. The study of meaning theories, so to speak, provides the *data* for the theory of meaning. In the present Chapter we shall mainly treat meaning theories and some of their connections with (technical) proof theory and, consequently, we shall only touch on the theory of meaning in passing. (On the other hand the whole Chapter can be viewed as a contribution to the theory of meaning.)

There is, since Frege, a large consensus that the sentence, rather than the word, is the primary bearer of (linguistic) meaning. The sentence is the least unit of language that can be used to *say* anything. Thus the theory of meaning directs that sentence-meaning is to be central in meaning theories and that word-meaning is to be introduced derivatively: the meaning of a word is the way in which the word contributes to the meaning of the sentences in which it occurs. It is natural to classify the sentences of a language according to the sort of linguistic act a speaker would perform through an utterance of the sentence in question, be it an assertion, a question or a command. Thus, in general, the meaning of a sentence seems to comprise (at least) two elements, because to know the meaning of – in order to understand an utterance of – the sentence in question one would have to know, first to what category the sentence belongs, i.e., one would have to know what sort of linguistic act that would be performed through an utterance of the sentence, and secondly one would have to know the *content* of the act.

This diversity of sentence-meaning, together with the idea that word-meaning is to be introduced derivatively (as a way of contributing to sentence-meaning), poses a certain problem for the putative meaning-theorist. If sentences from different categories have different *kinds* of meaning, it appears that the meaning of a word will vary according to the category of the sentences in which it occurs: uniform word-meanings are ruled out. But this is unacceptable as anyone familiar with a dictionary knows. The word 'door', say, has the same meaning in the three sentences 'Is the door open?', 'The door is open.', and 'Open the door!'. This *prima facie* difficulty is turned into a tool for investigating what internal structure ought to be imposed on a satisfactory meaning theory.

A meaning theory will have to comprise at least two parts: the *theory of sense* and the *theory of force*. The task of the latter is to identify the sort of act performed through an utterance of a sentence and the former has to specify the content of the acts performed. In order to secure the uniformity of word meaning the theory of sense has to be formulated in terms of some one *key concept*, in terms of which the content of all sentences is to be given, and the theory of force has to provide uniform, general, principles relating speech act to content. The meaning of a word is then taken as the way in which the word contributes to the content of the sentences in which it occurs (as given by the key concept in the theory of sense).

The use of such a notion of key concept also allows the meaning theories to account for certain (iterative) unboundedness-phenomena in language, e.g., that whenever *A* and *B* are understood sentences, then also '*A* and *B*' would appear to be meaningful. This is brought under control in the meaning theory by expressing the condition for the application of the key concept *P* to '*A* and *B*' in terms of *P* applied to *A* and *P* applied to *B*.

The most popular candidate for a key concept has undeniably been *truth*: the content of a sentence is given by its 'truth-condition'. One can, indeed, find many philosophers who have subscribed to the idea that meaning is to be given in terms of truth. Examples would be Frege, Wittgenstein, Carnap, Quine and Montague. It is doubtful, however, if they would accept that the way in which truth serves to specify meaning is as a key concept in a meaning theory (that is articulated into sense and force components respectively). Such a conception of the relation between meaning and truth has been advocated by Donald Davidson, who, in an important series of papers, starting with [1967], and now conveniently collected in his [1984], has proposed and developed the idea that meaning is to be studied via meaning theories. Davidson is quite explicit on the role of truth. It is going to take its rightful place within the meaning theory in the shape of a truth theory in the sense

of Tarski [1956], Ch. VIII. Tarski showed, for a given formal language L , how to define a predicate ' $\text{True}_L(x)$ ' such that for every sentence S of L it is provable from the definition that

$$(1) \quad \text{True}_L(\bar{S}) \text{ iff } f(S).$$

Here ' \bar{S} ' is a *name* of, and $f(S)$ a *translation* of, the object-language sentence S in the language of the meta-theory (= the theory in which the truth definition is given and where all instances of (1) must hold). Using the concept of meaning (in the guise of 'translation' from object-language to meta-language) Tarski gave a precise definition of what it is for a sentence of L to be true. Davidson reverses the theoretical priorities. Starting with a *truth theory* for L , that is a theory the language of which contains $\text{True}_L(x)$ as a primitive, and where for each sentence S of L

$$(2) \quad \text{True}_L(\bar{S}) \text{ iff } p$$

holds for some sentence p of the language of the truth theory, he wanted to extract meaning from truth. Simply to consider an arbitrary truth theory will not do to capture meaning, though. It is certainly true that

$$(3) \quad \overline{\text{Snow is white is true-in-English}} \text{ iff snow is white}$$

but, unquestionably and unfortunately, it is equally true that

$$(4) \quad \overline{\text{Snow is white is true-in-English}} \text{ iff grass is green}$$

and the r.h.s. of (4) could not possibly by any stretch of imagination be said to provide even a rough approximation of the meaning of the English sentence

Snow is white.

Furthermore, a theory that had all instances of (2) as axioms would be unsatisfactory also in that it used infinitely many unrelated axioms; the theory would, it is claimed, be 'unlearnable'.

Thus one might attempt to improve on the above simple-minded (2) by considering truth theories that are formulated in a meta-language that contains the object-language and that give their '*T*-theorems' (the instances of (2)), not as axioms, but as derivable from homophonic recursion clauses, e.g.,

$$(5) \quad \text{for all } \bar{A} \text{ and } \bar{B} \text{ of } L,$$

$$\text{True}_L(\overline{A \text{ and } B}) \text{ iff } \text{True}_L(\bar{A}) \text{ and } \text{True}_L(\bar{B})$$

and

- (6) for all \bar{A} of L ,
 $\text{True}_L(\overline{\text{not-}\bar{A}})$ iff $\text{not-True}_L(\bar{A})$.

Here one uses the word mentioned in the sentence on the l.h.s. when giving the condition for its truth on the r.h.s.; cf. the above remarks on the iterative unboundedness phenomena.

The treatment of quantification originally used Tarski's device of 'satisfaction relative to assignment by sequences', where, in fact, one does not primarily recur on truth, but on satisfaction, and where truth is defined as satisfaction by all sequences. The problem which Tarski solved by the use of the sequences and the auxiliary notion of satisfaction was how to capture the right truth condition for 'everything is A ' even though the object-language does not contain a name for everything to be considered in the relevant domain of quantification. Another satisfactory solution, which goes back to Frege, would be to use quantification over finite extensions L^+ of L by means of new names. The interested reader is referred to Evans [1977], Section 2 or to Davies [1981], Chapter VI for the (not too difficult) technicalities. A very extensive and careful canvassing of various alternative approaches to quantificational truth-theories is given by Baldwin [1979]. If we by-pass the problem solved by Tarski and consider, say, the language of arithmetic, where the problem does not arise as the language contains a numeral for each element of the intended domain of quantification, the universal-quantifier clause would be

- (7) for all \bar{A} of L ,
 $\text{True}_L(\overline{\text{for every number } x, A(x)})$ iff for every numeral \bar{k} ,
 $\text{True}_L(A(\bar{k}/x))$

(here ' $A(\bar{k}/x)$ ' indicates the result of substituting the numeral k for the variable x .)

Unfortunately it is still not enough to consider these homophonic, finitely axiomatized truth theories in order to capture meaning. The basic clauses of a homophonic truth theory will have the form, say,

- (8) for any name \bar{t} of L ,
 $\text{True}_L(\overline{t \text{ is red}})$ iff whatever t refers to is red.

If we now change this clause to

- (9) for any \bar{t} in L ,
 $\text{True}'_L(\overline{t \text{ is red}})$ iff whatever t refers to is red and grass is green

and keep homophonic clauses for True'_L with respect to 'and', 'not', etc., the result will still be a finitely axiomatized and correct ('true') truth theory for L . (We could equally well have chosen any other true contingent sentence instead of 'grass is green'.) Seen from the perspective of 'real meaning' the truth condition of the primed theory is best explained as

$$(10) \quad \text{True}'_L(\bar{S}) \text{ iff } S \text{ and grass is green.}$$

The fact that a true, finitely axiomatized, homophonic truth-theory does not necessarily provide truth conditions that capture meaning was first observed by Foster and Loar in 1976. Various remedies and refinements of the original Davidsonian programme have been explored. We shall briefly consider an influential proposal due to John McDowell [1976, 1977, 1978].

The above attempts to find a meaning theory via truth start with a (true) truth theory and go on to seek further constraints that have to be imposed in order to capture meaning. McDowell, on the other hand, reverses this strategy and starts by considering a satisfactory theory of sense. Such a theory has to give content-ascriptions to the sentences \bar{S} of the language L , say in the general form

$$(11) \quad \bar{S} \text{ is } Q \text{ iff } p,$$

where p is a sentence of the meta-language that gives the content of \bar{S} , and, furthermore, the theory has to interact with a theory of force in such a way that the interpreting descriptions, based on the contents as assigned in (11), do in fact make sense of what speakers say and do when they utter sentences containing \bar{S} . A meaning theory, and thus also its theory of sense, is part of an overall theory of understanding, the task of which is to make sense of human behaviour (and not just these speech-acts). If the theory of sense can serve as a content-specifying core in such a general theory, then (11) guarantees that the predicate Q is (co-extensional with) truth. But not only that is true; the pathological truth-theories that were manufactured for use in the Foster-Loar counter-examples are ruled out from service as theories of sense because their use would make the meaning theory issue incomprehensible, or outright false, descriptions of what people do. A theory of sense which uses a pathological truth-theory does not make sense. Thus we see that while an adequate theory of sense will be a truth theory, the opposite is false: not every truth theory for a language will be a theory of sense for the language.

In conclusion of the present section let us note the important fact that the Tarski homophonic truth-theories are completely neutral with respect to the underlying logic. The T -theorems are derivable from the basic

homophonic recursion clauses using intuitionistic logic only (in fact even minimal logic will do).

No attempt has been made in the present section to achieve either completeness or originality. The very substantial literature on the Davidsonian programme is conveniently surveyed in two texts, Platts [1979] and Davies [1981], where the latter pays more attention to the (not too difficult) technicalities. Many of the important original papers are included in Evans and McDowell [1976], with an illuminating introduction by the editors, and Platts [1980], while mention has already been made of Davidson's [1984] collection of essays.

2. INTERMEZZO: CLASSICAL TRUTH AND SEQUENT CALCULI

(Intended for readers with some knowledge of the method 'semantic tableaux', cf. Hodges, Chapter I.1, Section 6 or Sundholm, Chapter I.2, Section 3 of the *Handbook*.)

It is by now well-known that perhaps the easiest way to prove the completeness of classical predicate logic is to search systematically for a counter-model (or, more precisely, a falsifying 'semi-valuation', or 'model set') to the formula, or sequent, in question. This systematic search proceeds according to certain rules which are directly read off as necessary conditions from the relevant semantics. For instance, in order to falsify $\forall xA(x) \rightarrow B$, one needs to verify $\forall xA(x)$ and falsify B , and in order to verify $\forall xA(x)$ one has to verify $A(t)$ for every t , etc. Thus the rules for falsification, in fact, also concern rules for verification and *vice versa* (consider verification of, e.g., $\neg B$), and for each logical operator there will be two rules regulating the systematic search for a counter-model, one for verification and one for falsification. These rules turn out to be identical with Gentzen's [1934-1935] left and right introduction rules for the same operators. In some cases the search needs to take alternatives into account, e.g., $A \rightarrow B$ is verified by falsifying A or verifying B . Thus one has two possibilities. The failure of the search along a possibility is indicated by that the rules would force one to assign both truth and falsity to one and the same formula. This corresponds, of course, to the axioms of Gentzen's sequent calculi. This method, where failure of existence of counter-models is equivalent to existence of a sequent calculus proof-tree, was discovered independently by Beth, Hintikka, Kanger, and Schütte in the 1950s and a brilliant exposition can be found in Kleene [1967], Chapter VI. whereas Smullyan [1968] is the canonical

reference for the various ways of taking the basic insight into account. Prawitz [1975] is a streamlined development of the more technical aspects which provides an illuminating answer to the question as to why the rules that generate counter-models turn out to be identical with the sequent calculus rules. There one also finds a good introduction to the notion of semi-valuation which has begun to play a role in recent investigations into the semantics of natural language (cf. van Benthem and van Eijck [1982] for an interesting treatment of the connection between recent work on 'partial structures' in the semantics of natural language and the more proof-theoretical notions that derive from the 'backwards' completeness proofs).

These semantical methods for proving completeness also lend themselves to immediate proof-theoretical applications. The *Cut-free* sequent calculus is complete, but cut is a sound rule. Hence it is derivable. A connection with the topic of our Chapter is forged by reversing these proof-theoretic uses of semantical methods. Instead of proving the completeness via semantics, one could start by postulating the completeness of a cut-free formalism, and *read off a semantics* from the left and right introduction rules. (Proof theory determines meaning.) Such an approach was suggested by Hacking [1979] in an attempt towards a criterion for logical constanhood. Unfortunately, his presentation is marred by diverse technical infelicities (cf. Sundholm [1981]), and the problem still remains open how to find a workable proposal along these lines.

3. DUMMETT'S ARGUMENT AGAINST A TRUTH-CONDITIONAL VIEW ON MEANING

In the present Section I attempt to set out one version of an argument due to Michael Dummett to the effect that truth cannot adequately serve as a key concept in a satisfactory meaning theory. Dummett has presented his argument in many places (cf. the note at the end of the Section) and the presentation I offer is not to be attributed to him. In particular, the emphasis on manifestation that can be found in the present version of Dummett's argument I have come to appreciate through the writings of Colin McGinn [1980] and Crispin Wright [1976]. Dummett's most forceful exposition is still his [1976], which will be referred to as "WTM2".

Dummett's views on the role and function of meaning theories are only in partial agreement with those presented in Section 1. The essential difference consists mainly in the strong emphasis on what it is to know a language that can be found in Dummett's writings, and as a consequence his meaning

theories are firmly cast in an epistemological mould: "questions about meaning are best interpreted as questions of understanding: a dictum about what the meaning of a sentence consists in must be construed as a thesis about what it is to *know* its meaning" (WTM2, p. 69). The task of the meaning theorist is to give a theoretical (propositional) representation of the complex practical capacity one has when one knows how to speak a language. The knowledge that a speaker will have of the propositions that constitute the theoretical representation in question will, in the end, have to be *implicit* knowledge. Indeed, one cannot demand that a speaker should be able to articulate explicitly those very principles that constitute the theoretical representation of his practical mastery. Thus a meaning theory that gives such a theoretical representation must also comprise a part that would state what it is to know the other parts implicitly.

The inner structure of a meaning theory that could serve the aims of Dummett will have to be different from the simple bipartite version considered in Section 1. Dummett's meaning theories are to be structured as follows. There is to be (ia) a core *theory of semantic value*, which states the condition for the application of the key concept to the sentences of the language, and, furthermore, there must be (ii) a *theory of force*, as before. In between these two, however, there must be (ib) a *theory of sense*, whose task it is to state what it is to know what is stated in the theory of semantic value, i.e., what it is to know the condition for the application of the key concept to a sentence. Thus the theory of sense in the proposals from Section 1 does not correspond to the theory of sense_D – 'D' for Dummett – but to the theory of semantic value. (The Fregean origin of Dummett's tripartite structure should be obvious. For further elaboration cf., e.g., his [1981].) The theory of sense_D has no matching part in the theories from Section 1. The corresponding match is as follows:

<i>Dummett</i>		<i>(Davidson-) McDowell</i>	
(ia)	Theory of semantic value (applies key concept to sentences)	(i)	Theory of sense
(ib)	Theory of sense _D (states what it is to know the theory of semantic value)		
(ii)	The theory of force	(ii)	The theory of force

This difference is what lies at the heart of the matter in the discussion between Dummett and McDowell of whether a theory of meaning ought to be 'modest' or 'fullblooded' (cf. McDowell [1977], Section X: should one demand that the meaning theory must give a link-up with practical capacities *independently of, and prior to, the theory of force?*).

One should also note here that the right home for the theory of sense_D is not quite clear. Here I have made it part of the meaning theory. It could perhaps be argued that a statement of wherein knowledge of meaning consists is something that had better be placed within a theory of meaning rather than in a meaning theory. Dummett himself does not draw the distinction between meaning theories and theories of meaning and one can, it seems to me, find traces of *both* notions in what Dummett calls a 'theory of meaning'.

Dummett's argument against the truth-theoretical conception of meaning makes essential use of the assumption that the meaning theories must contain a theory of sense_D, which Dummett explicates in terms of how it can be manifested: since the knowledge is implicit, possession thereof can be construed only in terms of how one manifests that knowledge. Furthermore, this implicit knowledge of meaning, or more precisely, of the condition for applying the key concept to individual sentences, must be *fully* manifested in use. This is Dummett's transformation of Wittgenstein's dictum that meaning is use. Two reasons can be offered (cf. McGinn [1980], p. 20). First, knowledge is one of many propositional attitudes and these are, in general, only attributed to agents on the basis of how they are manifested. Secondly, and more importantly, we are concerned with (implicit) knowledge of *meaning* and meaning is, *par excellence*, a vehicle of (linguistic) communication. If there were some components of the implicit knowledge that did not become fully manifest in use, they could not matter for communication and so they would be superfluous.

It was already noted above that the Tarskian truth-theories are completely neutral with respect to the logical properties of truth. What laws are obeyed is determined by the logic that is applied in the meta-theory, whereas the *T*-clauses themselves offer no information on this point. Dummett's argument is brought to bear not so much against Tarskian truth as against the possibility that the key concept could be 'recognition-transcendent'. Classical, *bivalent* truth is characterized by the law of bivalence that every sentence is either true or false independently of our capacity to decide, or find out, whichever is the case. Thus, in general, the truth-conditions will be such that they can obtain without us recognizing that they do. There are a number of critical cases which produce such undecidable truth-conditions. (It should be

noted that 'undecidable' is perhaps not the best choice here with its connotations from recursive function theory.) Foremost among these is undoubtedly quantification over infinite or unbounded domains. Fermat's last theorem and the Riemann hypothesis are both famous examples from mathematics and their form is purely universal $\forall xA(x)$, with decidable matrix $A(x)$. An existential example would be, say, 'Somewhere in the universe there is a little green stone-eater'. Other sorts of examples are given by, respectively, counterfactual conditionals and claims about sentience in others, e.g., 'Ronald Reagan is in pain'. A fourth class is given by statements about (remote) past and future time, e.g., 'A city will be built here in a thousand years', or 'Two seconds before Brutus stabbed Caesar thirty-nine geese cackled on the Capitol'.

The knowledge one has of how to apply the key concept cannot in its entirety be statable, explicit knowledge and so the theory of sense_D will have to state, for at least some sentences, how one manifests knowledge of the condition for applying the key concept to them, in ways other than stating what one knows explicitly. Let us call the class of these 'the non-statable fragment'. (Questions of the 'division of linguistic labour' may arise here. Is the fragment necessarily unique? Cf. McGinn [1980], p. 22).

Assume now for a *reductio* that bivalent, possibly recognition-transcendent, truth-conditions can serve as key concept in a (Dummettian) meaning theory. Thus the theory of sense_D has to state how one fully manifests knowledge of possibly recognition-transcendent truth-conditions. The 'possibly' can be removed: there are sentences in the non-statable fragment with undecidable truth-conditions. In order to see this, remember the four classes of undecidable sentences that were listed above. Demonstrably, undecidable sentences are present in the language, and they must be present already in the non-statable fragment, because "the existence of such sentences cannot be due solely to the occurrence of sentences introduced by purely verbal explanations: a language all of whose sentences were decidable would continue to have this property when enriched by expressions so introduced" (WTM2, p. 81). An objection that may be (and has been) raised here is that one could start with a decidable fragment, e.g., the atomic sentences of arithmetic and get the undecidability through addition of new sentence-operators such as quantifiers. That is indeed so, but is not relevant here, where one *starts with a larger language* that, as a matter of fact, contains undecidable sentences and then isolates a fragment within this language that also will have this property. Decidable sentences used for definitions could only provide decidable sentences and hence some of the sentences

of the full language would be left out. Also it is not permissible to speak of adding, say, the quantifiers as their nature is *sub judice*: the meaning of a quantifier is not something independent of the rest of the language but, like any other word, its meaning is the way it contributes to the meaning of the sentences in which it occurs.

Now the argument is nearly at its end. The theory of sense_D would be *incomplete* in that it could not state what it is to manifest fully implicit knowledge of the recognition-transcendent truth-condition of an undecidable sentence. If the theory attempted to do this, an observational void would exist without observational warrant. We, as theorists, would be guilty of theoretical slack in our theory, because we could never see the agents manifest their implicit knowledge in response to the truth-conditions obtaining (or not), because, *ex hypothesi*, they obtain unrecognizably. The agents, furthermore, could not see them obtain and so, independently of whether or not the theorist can see them response, they cannot manifest their knowledge in response to the truth-condition. (This is a point where the division of linguistic labour may play a role.)

Before we proceed it might be useful to offer a short schematic summary of Dummett's argument as set out above. (Page references in brackets are to WTM2.)

- (1) To understand a language is to have knowledge of meaning. (p. 69).
- (2) Knowledge of meaning must in the end be implicit knowledge. (p. 70).
- (3) Hence the meaning theory must contain a part, call it theory of sense_D , that specifies "in what having this knowledge consists, i.e., what counts as a manifestation of that knowledge." (pp. 70–71 and p. 127).
- (4) There are sentences in the language such that the speaker manifests his knowledge of their meaning in ways other than stating the meaning in other words. (The non-statable fragment is non-empty.) (p. 81).
- (5) Assume now that bivalent truth can serve as key concept. Bivalent truth-conditions are sometimes undecidable and hence recognition-transcendent. (p. 81).
- (6) Already in the non-statable fragment there must be sentences with recognition-transcendent truth-conditions. (p. 81).
- (7) Implicit knowledge of recognition-transcendent truth-conditions cannot be manifested, and so the theory of sense_D is incomplete. (p. 82).

Supplementary notes concerning the argument:

(a) Dummett's argument is quite general and does not rest at all on any specific features of the language concerned. When it is applied to a particular area of discourse, or for a particular class of statements, it will lead to a metaphysical anti-realism for the area in question. Many examples of this can be found in Dummett's writings. Thus [1975] and [1977] both develop the argument within the philosophy of mathematics. The intuitionistic criticism of classical reasoning, and the ensuing explanations of the logical constants offered by Heyting, provided the main inspiration for Dummett's work on anti-realism. It should be stressed, however, and as is emphasized by Dummett himself in [1975], that the semantical argument in favour of a constructivist philosophy of mathematics is very far from Brouwer's own position.

In Dummett [1969] another one of the four critical classes of sentences is studied, viz. those concerning time, and in WTM2, Section 3, a discussion of counterfactual conditionals can be found, as well as a discussion of certain reductionist versions of anti-realism. They arise when the truth of statement A is reduced to the (simultaneous) truth of a certain possibly infinite class of reduction-sentences M_A . If it so happens that the falsity of the conjunction $\bigwedge M_A$ does not entail the truth of the conjunction $\bigwedge M_{\neg A}$, then bivalence will fail for the statement A . Examples of such reductionist versions of anti-realism can be found in phenomenalist reductions of material objects or of sentence in others.

(b) It should be noted that Dummett's anti-realism, while verificationist in nature, must not be conflated with logico-empiricist verificationism. With a lot of simplification the matter can be crudely summarized by noting that for the logical empiricists classical logic was sacrosanct and certain sentences have non-verifiable classical truth-conditions. Hence they have no meaning. Dummett reverses this reasoning: obviously meaningful sentences have no good meaning if meaning is construed truth-conditionally. Hence classical meaning-theories are wrong.

(c) As one should expect, Dummett's anti-realist argument has not been allowed to remain uncontroverted. John McDowell has challenged the demand that the meaning theories should comprise a theory of sense_D. In his [1977] and [1978] the criticism is mainly by implication as he is there more concerned with the development of the positive side of his own 'modest' version of a meaning theory, whereas in [1981] he explicitly questions the cogency of Dummett's full-blooded theories. McDowell's [1978a] is an answer to Dummett's [1969], and McDowell in his turn has found a critic in Wright [1980a].

Colin McGinn has been another persistent critic of Dummett's anti-realism and he has launched counter-arguments against most aspects of Dummett's position, cf. e.g., his [1980], [1979] and [1982].

Crispin Wright [1982] challenges Dummett by observing that a Strict Finitist can criticize a Dummettian constructivist in much the same way as a constructivist uses Dummett's argument against a mathematical Platonist and so the uniquely privileged position that is claimed for constructivism (as the only viable alternative to classical semantics) is under pressure.

(d) Another sort of criticism is offered by Dag Prawitz [1977, 1978], who, like Wright, is in general sympathy with large parts of Dummett's meaning-theoretical position. Prawitz questions the demand for *full* manifestation and suggests that the demand for a theory of sense_D be replaced by an *adequacy condition* on meaning theories *T*:

if *T* is to be adequate, it must be possible
to derive in *T* the implication
if *P* knows the meaning of *A*, then *P* shows behaviour *B_A*.

Prawitz [1978], p. 27

(Here "*B_A*" is a kind of behaviour counted as a sign of grasping the meaning of *A*.)

The difference between this adequacy criterion and the constraints that McDowell imposes on his modest theories is not entirely clear to me. Only if the behaviour is to be shown before, and independently of, the theory of force (whose task it is to issue just the interpreting descriptions that tell what behaviour was exhibited by *P*) could something like a modification of Dummett's argument be launched and even then it does not seem certain that the desired conclusion can be reached.

(e) In the presentation of Dummett's argument I have relied solely on WTM2. The anti-realist argument can be found in many places though, e.g., [1973], Chapter 13, [1969], [1975] and [1975a] as well as the recent [1982]. It should be noted that Dummett often, cf. e.g., [1969], lays equal or more stress on the acquisition of knowledge rather than its manifestation. Most of the articles mentioned are conveniently reprinted in TE.

Wright [1981], a review of TE, gives a good survey of Dummett's work. Similarly, in his book [1980] Wright offers extensive discussion of anti-realist themes.

The already mentioned McGinn [1980] and Prawitz [1977], while not in entire agreement with Dummett, both give excellent expositions of the

basic issues. It is a virtually impossible task to give a complete survey of the controversy around Dummett's anti-realist position. In recent years almost every issue of the *Journal of Philosophy*, *Mind* and *Analysis*, as well as the *Proceedings of the Aristotelean Society*, contains material that directly, or indirectly, concerns itself with the Dummettian argument.

4. PROOF AS A KEY CONCEPT IN MEANING THEORIES

As was mentioned above the traditional intuitionistic criticism of classical mathematical reasoning, cf. e.g., van Dalen [this volume], was an important source of inspiration for Dummett's anti-realist argument and it is also to intuitionism that he turns in his search for an alternative key-concept to be used in the meaning theories in place of the bivalent, recognition-transcendent truth-conditions.

The simplest technical treatment of the truth-conditions approach to semantics is undoubtedly provided by the standard truth-tables (which, of course, are incorporated in the Tarski-treatment for, say, full predicate logic) and it is the corresponding constructive 'proof-tables' of Heyting that offer a possibility for Dummett's positive proposal. Heyting's explanations of the logical constants, cf. his [1956], Chapter 7, and [1960], can be set out roughly as follows;

A proof of the proposition	is given by
$A \wedge B$	a proof of A and a proof of B
$A \vee B$	a proof of A or a proof of B
$A \rightarrow B$	a method for obtaining proofs of B from proofs of A
\perp	nothing
$\forall x \in D A(x)$	a method which for every individual d in D provides a proof of $A(d)$
$\exists x \in D A(x)$	an individual d in D and a proof of $A(d)$.

There are various versions of the above table of explanations, e.g., the one offered by Kreisel [1962], where 'second clauses' have been included in the explanations for implication and universal quantification to the effect that one has to include also a proof that the methods really have the properties required in the explanations above. The matter is dealt with at length in Sundholm [1983], where an attempt is made to sort out the various issues

involved and where extensive bibliographical information can be found, cf. also Section 7 below on the type theory of Martin-Löf.

In the above explanations the meaning of a proposition is given by its 'proof condition' and, as was emphatically stressed by Kreisel [1962], in some sense, 'we recognize a proof when we see one'. Thus it seems that the anti-realistic worries of Dummett can be alleviated with the use of proof as a key-concept in meaning theories. (I will return to this question in the next section.) Independently of the desired immunity from anti-realist strictures, however, there are a number of other points that need to be taken into account here.

First among these is a logical gem invented by Prior [1960]. In the Heyting explanations the meaning of a proposition is given by its proof-condition. Conversely, does every proof-condition give a proposition? A positive answer to this question appears desirable, but the notion 'proof-condition' needs to be much more elucidated if any headway is to be made here. Prior noted that if by 'proof-conditions' one understands 'rules that regulate deductive practice' then a negative answer is called for. Let us introduce a proposition-forming operator, or connective, 'tonk' by stipulating that its deductive practice is to be regulated by the following Natural Deduction rules (I here alter Prior's rules inessentially):

tonk I	$\frac{A}{A \text{ tonk } B}$	$\frac{B}{A \text{ tonk } B}$
tonk E	$\frac{A \text{ tonk } B}{A}$	$\frac{A \text{ tonk } B}{B}$

As Prior observes one then readily proves false conclusion from true premises by means of first tonk I and then tonk E. In fact, given these two rules *any two propositions are logically equivalent* via the following derivation:

$$\begin{array}{c}
 \text{(tonk I)} \frac{A^1}{A \text{ tonk } B} \quad \frac{B^2}{A \text{ tonk } B} \text{(tonk I)} \\
 \hline
 \frac{B}{A \leftrightarrow B} \quad \frac{A}{1, 2 (\leftrightarrow I)}
 \end{array}$$

Thus tonk leads to extreme egalitarianism in the underlying logic: from a logical point of view there is only one proposition. This is plainly absurd and something has gone badly wrong. Hence it is clear (and only what could be expected) that some constraints are needed for how the proof-conditions are to be understood; 'rules regulating deductive practice' is simply too broad.

There is quite a literature dealing with tonk and the problems it causes: Stevenson [1961], Wagner [1981], Hart [1982], and, perhaps most importantly from our point of view, Belnap [1962], more about which below. The relevance of the tonk-problem for our present interests, was as far as I know, first noted by Dummett [1973], Chapter 13.

A second point to consider is the so-called *paradox of inference*, cf. Cohen and Nagel [1934], pp. 173–176. This ‘paradox’ arises because of the tension between (a) the fact that a correct application of a logically valid inference seems to presuppose that the truth of the conclusion is already contained in the truth of the premises, and (b) the fact that logical inference is a way to gain ‘new’ knowledge. Cohen and Nagel formulate it thus:

If in an inference the conclusion is not contained in the premise, it cannot be valid; and if the conclusion is not different from the premises, it is useless; but the conclusion cannot be contained in the premises and also possess novelty; hence inferences cannot be both valid and useful.

[1934], p. 173.

So there is a tension between the legitimacy (the validity) and the utility of an inference, and one could perhaps reformulate the question posed by the ‘paradox’ as: How can logic function as a useful epistemological tool? For an inference to be legitimate, the process of recognizing the premises as true must already have accomplished what is needed for the recognition of the truth of the conclusion, but if it is to be useful the recognition of the truth of the conclusion does not have to be present when the truth of the premises is ascertained. This is how Dummett poses the question in [1975a].

How does one use reasoning to gain new truths? By starting with known premises and drawing further conclusions. In most cases the use of valid inference has very little to do with how one would normally set about to verify the truth of something. For instance, the claim that I have seven coins in my pocket is best established by means of counting them. It would be possible, however, to deduce this fact from a number of diverse premises and some axioms of arithmetic. (The extra premises would be, say, that I began the day with a £50 note, and I have then made such and such purchases for such and such sums, receiving such and such notes and coins in return, etc.) This would be a highly *indirect* way in comparison with the straightforward counting process. The utility of logical reasoning lies in that it provides indirect means of learning the truth of statements. Thus in order to account for this usefulness it seems that there must be a gap between

the most direct ways of learning the truth and the indirect ways provided by logic. If we now explain meaning in terms of proof, it seems that we close this gap. The direct means, given directly by the meaning, would coincide, so to speak, with the indirect means of reasoning. The indirect means have then been made a part of the direct means of reasoning. (One should here compare the difference between direct and indirect means of recognizing the truth with the solution to the 'paradox' offered by Cohen and Nagel [1934] that is formulated in terms of a concept called 'conventional meaning'.)

The constraints we seek on our proof-explanations thus should take into account, on the one hand, that one must not be too liberal as witnessed by tonk, and, on the other hand, one must not make the identification between proof and meaning so tight that logic becomes useless.

Already Belnap [1962] noted what was wrong with tonk from our point of view. The (new) deductive practice that results from adding *tonk* with its stipulative rules, is not *conservative* over the old one. Using Dummett's [1975] terminology, there is no *harmony* between the grounds for asserting, and the consequences that may be drawn from, a sentence of the form *A tonk B*. The introduction and elimination rules must, so to speak, match, not just in that each connective has introduction and elimination rules, but also in that they must not interfere with the previous practice. Hence it seems natural to let one of the (two classes of) rules serve as meaning-giving and let the other one be chosen in such a way that it(s members) can be justified according to the meaning-explanation. Such a method of proceeding would also take care of the 'paradox' of inference: one of the two types of rules would now serve as the direct, meaning-given (because meaning-giving!) way of learning the truth and the other would serve to provide the indirect means (in conjunction with other justified rules, of course).

The introduction rules are the natural choice for our purpose, since they are *synthesizing* rules; they explain how a proof of, say $A \text{ } \$ \text{ } B$, can be formed in terms of given proofs of *A* and of *B*, and thus some sort of compositionality is present (which is required for a key concept). Tentatively then, the meaning of a sentence is given by what counts as a *direct* (or *canonical*) *proof* of it. Other ways of formulating the same explanation would be to say that the meaning is given by the *direct grounds for asserting*, or by what counts as a *direct verification* of, the sentence in question. An (indirect) proof of a sentence would be a method, or program, for obtaining a direct proof.

In order to see that a sentence is true one does not in general have to

produce the direct grounds for asserting it and so the desired gap between truth and truth-as-established-by-the-most-direct-means is open. Note that one could still say that the meaning of a sentence is given by its truth-condition, although the latter, of course, has to be understood in a way different from that of bivalent, and recognition-transcendent, truth: if a sentence is true it is possible to give a proof of it and this in turn can be used to produce a *direct proof*. Thus in order to explain what it is for a sentence to be true one has to explain what a direct proof of the sentence would be and, hence, one has to explain the meaning of the sentence in order to explain its truth-condition.

All of this is highly programmatic and it remains to be seen if, and how, the notion of *direct* (canonical) *proof* (verification, ground for asserting) can be made sense of also outside the confined subject-matter of mathematics. In the next Section I shall attempt to spell out the Heyting explanations once again, but now in a modified form that closely links up with the discussion in the present Section and with the so-called normalization theorems in Natural Deduction style proof theory.

5. THE MEANING OF THE LOGICAL CONSTANTS AND THE SOUNDNESS OF PREDICATE LOGIC

In the present section, where knowledge of Natural Deduction rules is presupposed, we reconsider Heyting's explanations and show that the introduction and elimination rules are sound for the intended meaning.

Thus we assume that A and B are *meaningful sentences*, or *propositions*, and, hence that we know what proofs (and direct proofs) are for them.

The *conjunction* $A \wedge B$ is a proposition, such that a canonical proof of $A \wedge B$ has the form

$$\begin{array}{cc} D_1 & D_2 \\ A & B \\ \hline A \wedge B \end{array}$$

where D_1 and D_2 are (not necessarily direct) proofs of A and B , respectively. On the basis of this meaning-explanation of the proposition $A \wedge B$, the rule ($\wedge I$) is seen to be valid. We have to show that whenever the two premises A and B are true, then so is $A \wedge B$. When A and B are true, they are so on the basis of proof and hence there can be found two proofs D_1 and D_2 respectively of A and B . These proofs can then be used to obtain a canonical proof of $A \wedge B$, which therefore is true.

Consider the elimination rule ($\wedge E$), say, $\frac{A \wedge B}{B}$, and assume that $A \wedge B$ is true. We have to see that B is true. $A \wedge B$ is true on the basis of a proof D , which by the above meaning-explanation can be used to obtain a canonical proof D_3 of the form specified above. Thus D_2 is a proof of B and thus B is true.

Next we consider the *implication* $A \rightarrow B$, which is a proposition that is true if B is true on the assumption that A is true. Alternatively we may say that a canonical proof of $A \rightarrow B$ has the form

$$\frac{\begin{array}{c} A^1 \\ D \\ B \end{array}}{A \rightarrow B^1}$$

where D is a proof of B using the assumption A . Again, the introduction rule ($\rightarrow I$) is sound, since what has to be shown is that *if* B is true on the hypothesis that A is true, *then* $A \rightarrow B$ is true. But this is directly granted by the meaning explanation above. For the elimination rule we consider

$$\frac{A \rightarrow B \quad A}{B}$$

and suppose that we have proofs D_1 and D_2 of respectively $A \rightarrow B$ and A . As D_1 is a proof it can be used to obtain a canonical proof D_3 and thus we can find a hypothetical proof D of B from A . But then

$$\begin{array}{c} D_2 \\ A \\ D \\ B \end{array}$$

is a proof of B and thus B is true and ($\rightarrow E$) is a valid rule.

The *disjunction* $A \vee B$ is a proposition, with canonical proofs of the forms

$$\frac{\begin{array}{c} D_1 \\ A \end{array}}{A \vee B} \quad \text{and} \quad \frac{\begin{array}{c} D_2 \\ B \end{array}}{A \vee B}$$

where D_1 and D_2 are proofs of respectively A and B . The introduction rules are immediately seen to be valid, since they produce canonical proofs of their

true premise. For the elimination rule, we assume that $A \vee B$ is true, that C is true on assumption that A is true, and that C is true on assumption that B is true. Thus there are proofs D_1 , D_2 and D_3 of, respectively $A \vee B$, C and C , where the latter two proofs are hypothetical, depending on respectively A and B . The proof D_1 can be used to obtain a canonical proof D_4 of $A \vee B$ in one of the two forms above, say the right, and so D_4 contains a subproof D_5 , that is a proof of B . Then we readily find a proof of C by combining D_5 with the hypothetical D_3 to get a proof of C , which thus is a true proposition.

The *absurdity* \perp is a proposition which has *no* canonical proof. We have to see that the rule $\frac{\perp}{A}$ is valid. Thus, we have to see that whenever the proposition \perp is true, then also A is true. But \perp is never true, since a proof of \perp could be used to obtain a *canonical* proof of \perp and by the explanation above there are no direct proofs of \perp .

The *universal quantification* $(\forall x \in M)A(x)$ is a proposition such that its canonical proofs have the form

$$\frac{\begin{array}{c} x \in M^1 \\ D \\ A(x) \end{array}}{(\forall x \in M)A(x)^1}$$

that is, the proof D of the premise is a hypothetical, free-variable, proof of $A(x)$ from the assumption that $x \in M$. Again the introduction rule is valid, since if $A(x)$ is true on the hypothesis that $x \in M$, there can be found a hypothetical proof of $A(x)$ from assumption $x \in M$, and thus we immediately obtain a canonical proof of $(\forall x \in M)A(x)$. For the elimination rule $(\forall E)$ consider

$$\frac{(\forall x \in M)A(x) \quad d \in M}{A(d)}$$

and suppose that the premises are true. Thus proofs D_1 and D_2 of, respectively, $(\forall x \in M)A(x)$ and $d \in M$, can be found. As D_1 is a proof it can be used to obtain a direct proof of its conclusion, and hence we can extract a hypothetical proof D_3 of $A(x)$ from assumption $x \in M$. Combining D_2 with the free-variable proof D_3 gives a proof

$$\begin{array}{ll} D_2 & ('D_3(d/x)' \text{ indicates the} \\ d \in M & \text{result of substituting} \\ D_3(d/x) & d \text{ for } x \text{ in } D_3.) \\ A(d) & \end{array}$$

of $A(d)$, so the rule $(\forall E)$ is sound.

Finally, the *existential quantification* $(\exists x \in M)A(x)$ is a proposition such that its canonical proofs have the $(\exists I)$ form

$$\frac{\begin{array}{c} D_1 \\ A(d) \end{array} \quad \begin{array}{c} D_2 \\ d \in M \end{array}}{(\exists x \in M)A(x)}$$

Again the introduction rule is immediately seen to be valid as it produces canonical proofs of its conclusion from proofs of the premises. For the elimination rule $(\exists E)$ consider the situation that $(\exists x \in M)A(x)$ is true, and that C is true on the assumptions that x is in M and $A(x)$ is true. Thus there can be found a proof D_3 of $(\exists x \in M)A(x)$ and a hypothetical free-variable proof D_4 of C from hypotheses $x \in M$ and $A(x)$. The proof D_3 can be used to obtain a canonical proof of the form above, and combining the proofs D_1 and D_2 with the hypothetical free-variable proof D_4 we obtain a proof of C :

$$\frac{\begin{array}{c} D_2 \\ d \in M \end{array} \quad \begin{array}{c} D_1 \\ A(d) \end{array}}{D_4(d/x)} \\ C$$

Thus the rules of the intuitionistic predicate logic are all valid; no corresponding validation is known for, say, the classical law of Bivalence $A \vee \neg A$ where $\neg A$ is defined as $A \rightarrow \perp$.

The above treatment has been less precise and complete than would be desirable owing to limitations of space. First, questions of syntax have been left out especially where the quantifier rules are concerned, and secondly a whole complex of problems that arises from the fact that we need to know that $A(x)$ is a proposition for any x in M in order to know that, say, $(\forall x \in M)A(x)$ is a proposition has been ignored. The interested reader is referred to the type theory of Martin-Löf [1984] for detailed consideration and careful treatment of (analogues to) these and other lacunae, e.g., how to treat atomic sentences in our presentation.

The above explanations of why the rules of predicate logic are valid all follow the same pattern. The introduction rules are immediately seen to be valid, since canonical proofs are given introductory form. The elimination rules are then seen to be valid by noting that the introduction and elimination rules have the required harmony. The canonical grounds for asserting a sentence do contain sufficient grounds also for the consequences that may

be drawn via the elimination rules for the sentence in question. Thus, in fact, we have here made use of the *reduction steps* first isolated and used by Prawitz [1965, 1971], in his proofs of the normalization theorems for Natural Deduction-style formalizations.

Prawitz has in a long series of papers [1973, 1974, 1975, 1978, and 1980] been concerned to use this technical insight for meaning-theoretical purposes. His main concern, however, has been to give an explication of the notion of valid argument rather than to give direct meaning explanations in terms of proof. In the presentation here, which is inspired by Martin-Löf's meaning-explanations for his type theory, I have been more concerned with the task of giving constructivistic meaning-explanations while relying on the standard explication of validity as preservation of truth for a justification of the standard rules of inference.

One should, however, stop to consider the extent to which the above explanations constitute a meaning theory in the sense of Section 1 above. In particular, in Section 4 a promise was given to return to question of decidability. Is it in fact true that the notion of proof is decidable? On our presentation at least this much is true: if we already have a proof it is decidable if it is in canonical form. As to the general question I would be inclined to think that the notion of proof is *semi-decidable*, in that we recognize a proof when we see one, but when we don't see one that does not necessarily mean that there is no proof there. One can compare the situation with understanding a meaningful sentence: we understand a meaningful sentence when we see (or hear!) one but if we don't understand that does not necessarily mean that there is nothing there to be understood. Failure to understand a meaningful sentence seems parallel to failure to follow, or grasp, a proof. Such a position, then, would not make the 'proof-condition' recognition-transcendent; when it obtains it can be seen to obtain, but when it is not seen to obtain no judgement is given (unless, of course, it is seen not to obtain). Apart from the question of decidability, an important difference is that in explanations such as the above there is no mention of implicit knowledge and the like. It seems correct to speak of a theoretical representation of a (constructivistic) deductive practice, but it seems less natural to say that these explanations are known to everyone who draws logical inferences.

We used the notion of canonical proof as a key concept in order to provide the explanations, and in the literature one can find a number of alternatives as to how one ought to specify these, cf. the papers by Prawitz listed above. In particular, one might wish to insist that all parts of a canonical proof

should also be canonical (as is the case with the so-called normal derivations obtained by Prawitz in his normalization theorem [1971]). The choice I opted for here was motivated by, first, the success of the meaning-explanations of Martin-Löf in his type theory and, secondly, the fact that in Hallnäs [1983] a successful normalization of strong set-theoretic systems is carried out using an analogous notion of normal derivation. (Tennant's [1982] and his book [1978] are also interesting to the set-theoretically curious; in the former a treatment of the paradoxes is offered along Natural Deduction lines, and the latter contains a neat formulation of the rules of set theory.)

Finally we should note that the explanations offered here have turned the formal system into an *interpreted formal system* (modulo not inconsiderable imprecision in the formulation of syntax and explanations). This is the main reason for the avoidance of Greek letters in the present Chapter.

6. QUESTIONS OF COMPLETENESS

In Section 5 the meaning of the logical constants was explained and the standard deductive practice justified. In the case of classical, bivalent logic we know that the connectives \wedge , \vee and \neg are complete in that any truth-function can be generated from them. Does the corresponding property hold here? Clearly the answer is dependent on how the canonical proofs may be formed. It was shown by Prawitz [1978] and, independently of him, by Zucker and Tragesser [1978] that if we restrict ourselves to purely schematic means for obtaining canonical proofs (and for logical constants this does not seem unreasonable), then an affirmative answer is possible to the above question. As a typical example consider, e.g., this Sheffer-stroke (which of course makes sense constructively as well). This is given the introduction rule ($\mid I$)

$$\frac{\begin{array}{c} A^1 \dots B^2 \\ \vdots \\ \perp \end{array}}{A \mid B^{1,2}}$$

A definition using \wedge , \rightarrow and \perp is found by putting $A \mid B =_{\text{def}} A \wedge B \rightarrow \perp$. If there are more premise-derivations in the introduction rule (= the rule for how canonical proofs may be obtained) for each of these one will get an implication of the above sort and they are all joined together by conjunctions. (Here it is presupposed that the rules have only finitely many premises. This does not seem unreasonable.) Finally, if there are more introduction

rules than one, the conjunctions are put together into a disjunction. (Here it is presupposed that there are only finitely many introduction rules. Again this does not seem unreasonable).

Only one case remains, namely that there are no introduction rules. Then there are no canonical proofs to be found and we have got the absurdity. Thus the fragment based on \rightarrow , \wedge , \vee and \perp is complete. For further details refer to the two original papers above as well as Schroeder-Heister [1982]. It should be noted that by Hendry [1981] we know that $A \wedge B$ is equivalent also intuitionistically to $(A \leftrightarrow B) \leftrightarrow (A \vee B)$ and that $A \rightarrow B$ is equivalent to $B \leftrightarrow A \vee B$. Thus also \leftrightarrow , \vee and \perp are complete.

The standard elimination-rules ($\wedge E$) can be replaced by the following rule:

$$\frac{\begin{array}{c} A^1 \dots B^2 \\ \vdots \\ A \wedge B \end{array} \quad \begin{array}{c} C \\ \vdots \\ C \end{array}}{C} \quad 1,2$$

which rule seems quite well-motivated by the analogy with the introduction

rule $\frac{A \quad B}{A \wedge B}$: everything which can be derived from the two premises A and B used as assumptions can also be derived from $A \wedge B$ alone. The ($\vee E$) rule has exactly this general pattern and the intuitionistic absurdity rule is a degenerate case without minor premise C :

$$\frac{\perp}{C}$$

Only implication does not obey the above pattern. Here the premise of the introduction rule is not just a sentence, but a hypothetical judgement that B is true whenever A is true. Thus, we have a sort of rule as premise: from A go to B , in symbols $A \Rightarrow B$. If we may use such rules as *dischargable* assumptions, one can keep the standard pattern also for implication, viz.

$$\frac{\begin{array}{c} A \Rightarrow B^1 \\ \vdots \\ A \rightarrow B \end{array} \quad \begin{array}{c} C \\ \vdots \\ C \end{array}}{C} \quad 1$$

whereas if we try to do the same using implication for the arrow \Rightarrow , we end up with the triviality

$$\frac{A \rightarrow B \quad \begin{array}{c} A \rightarrow B^1 \\ \vdots \\ C \end{array}}{C} 1$$

which does not allow us to derive even *modus ponens*.

Using the rule with the higher level assumption $A \Rightarrow B$ one can derive $(\rightarrow E)$ as follows:

$$\frac{A \rightarrow B \quad \frac{A}{B} (A \Rightarrow B)^1}{B} 1$$

Given the use of the rule $A \Rightarrow B$ as an assumption, from premise A we can proceed to conclusion B , and the use of the major premise $A \rightarrow B$ allows to *discharge the use of the rule* $A \Rightarrow B$.

This type of higher-level assumptions was introduced by Schroeder-Heister [1981] and it is a most interesting innovation in Natural Deduction-formulations of logic, cf. also his [1982] and [1983]. The elimination rule that the Prawitz method gives to the Sheffer-stroke would be

$$\frac{A \wedge B \rightarrow \bot^1 \quad \begin{array}{c} \vdots \\ C \end{array}}{C} 1$$

which follows the above pattern, but uses implication and conjunction. With the Schroeder-Heister conventions the rule can be given as

$$\frac{(A, B \Rightarrow \bot)^1 \quad \begin{array}{c} \vdots \\ C \end{array}}{C} 1$$

In words, if C is true under the assumptions that we may go from the premises A and B to conclusion \bot , then C is a consequence of $A \mid B$ alone.

In Schroeder-Heister [1984] an extension of the above results is given and completeness is established also for the predicate calculus language.

The other question of completeness is also considered by Schroeder-Heister [1983]: is every valid inference derivable from the introduction and elimination rules? This question gets a positive answer, but the concept of validity is extremely restrictive, e.g., the rule
$$\frac{(A \wedge B) \wedge C}{A}$$
 is not a valid rule, cf. [1983], p. 374, which (given the concept of validity used in the present paper) it obviously must be. Thus I would consider the problem, first posed by Prawitz [1973], to establish the completeness of the predicate logic, for the present sort of meaning explanations, still to be open.

7. THE TYPE THEORY OF MARTIN-LÖF

Frege [1893], in the course of carrying out his logicist programme, designed a full-scale, completely formal language that was intended to suffice for mathematical practice. By today's standards, an almost unique feature of his attempt to secure a foundation of mathematics is that he uses an *interpreted* formal language for which he provides careful meaning explanations. The language proposed was, as we now know, not wholly successful, owing to the intervention of Russell's paradox. (The effects of the paradox on Frege's explanations of meaning are explored in Aczel [1980] and, from a different perspective, in Thiel [1975] and Martin [1982].) As the formal logic of Frege (and Whitehead-Russell) was transformed gradually into mathematical logic, notably by Tarski and Gödel, interest in the task of giving meaning explanations for interpreted formal languages faded out and after World War II the current distinction between syntax and (Tarskian, model-theoretic) semantics has become firmly entrenched.

The type theory of Martin-Löf [1975, 1982, 1984] represents a remarkable break with this tradition in that it returns to the original Fregean paradigm: interpreted formal language with careful explanations of meaning. Owing to limitations of space I shall not be able to give a detailed, precise description of the system here, (a task for which Martin-Löf [1984] uses close to a hundred pages), but will confine myself to trying to convey the basic flavour of the system.

A possible route to Martin-Löf's theory is through further examination of Heyting's explanations of the meaning of the logical constants. Our tentative semantics in Section 5 above made tacit use of a refinement of the explanations: the proof-tables do not give just proofs but *canonical*, or *direct*

proofs. A further refinement can be culled from Heyting's own writings. (In Sundholm [1983] a fairly detailed examination of Heyting's writings on this topic is offered.) According to Heyting, in order to prove a theorem one has to carry out certain constructions, 'die gewissen Bedingungen genügen', namely that it produces a mathematical object with certain specified properties, cf. e.g., his remarks on the proposition

"Euler's constant is rational"

in [1931], p. 113. In Martin-Löf's system, the proof-tables are extended to contain also the information about the objects that need to be constructed in order to establish the truth of the propositions in question. Thus, taking both refinements into account, the meaning of a proposition is explained by telling what a canonical object for the proposition would be. (A canonical object is not needed in order to assert the proposition; an object (method, program) that can be evaluated to canonical form is enough. For more details here, see Martin-Löf [1984].) In fact, according to Martin-Löf, one also has to tell when two such objects are equal. On the other hand, when one defines a set constructively, one has to specify what the canonical elements are and what it is for two elements of the set to be equal elements. Thus, the explanations of what propositions are and of what sets are, are completely analogous and Martin-Löf's system does not differentiate between the two notions.

In ordinary formal theories, that are formulated in the predicate calculus, the derivable objects are *propositions* (or, rather, they are well-formed formulae, i.e., the formalistic counterparts of propositions). This leads to certain difficulties for the standard formulation where logical inference is a relation between proposition. As was already observed by Frege, the correct formulation of *modus ponens* is

$$\frac{A \rightarrow B \text{ is true} \quad A \text{ is true}}{B \text{ is true}};$$

It is simply not correct to say that the proposition *B* follows from the propositions *A* \rightarrow *B* and *A*. What is correct is that the *truth* of the proposition *B* follows from the *truth* of *A* \rightarrow *B* and the *truth* of *A*. Thus *the premises and conclusions of logical inferences are not propositions but judgements as to the truth of propositions*. Furthermore, as Martin-Löf notes, that in order to keep the rules formal, one should also include the information that *A* and *B* are propositions in the premises of the rules, e.g.,

$$\frac{A \text{ is a prop.} \quad B \text{ is a prop.} \quad A \text{ is true}}{A \vee B \text{ is true}}$$

is how \vee -introduction should be set out. Therefore, as the premises of inferences are judgements, and remembering the identification of propositions and sets, one finds two main sorts of judgements in the theory, namely

$$(a) \quad A \text{ set} \quad ('A \text{ is a set}')$$

and

$$(b) \quad a \in A \quad ('a \text{ is an element of the set } A').$$

(In fact, there are two further forms of judgement, namely ' A is the same set as B ' and ' a and b are equal elements of the set A '.)

In accordance with the above discussion, (a) also does duty for ' A is a proposition' and (b) can also be read as 'the (proof-)object a is of the right sort for the proposition A , meets the condition specified by the proposition A '. This reading of (b) is, constructively, a longhand for the judgement '(the proposition) A (is) *true*', which is used whenever it is convenient to suppress the extra information contained in the proof-object. A third reading, deriving from Heyting and Kolmogorov, is possible, where (a) is taken in the sense ' A is a *task* (or *problem*)' and (b) in the sense ' a is a method for carrying out the task A (solving the problem A)'. When the task-aspect is emphasized, another reading would be ' a is a *program* that meets the *specification* A ' and the type-theoretical language of Martin-Löf [1982] has, owing to this possibility, had considerable influence as a programming language.

Some feeling for the interaction between propositions and proof-objects may be obtained through consideration of the simple example of *conjunction*. The *proposition* $A \wedge B$ (or *set* $A \times B$) is explained, on the assumption that A and B are propositions, by laying down that a canonical element of $A \times B$ is a pair (a, b) where $a \in A$ and $b \in B$. Thus the \times -introduction rule is correct:

$$\frac{a \in A \quad b \in B}{(a, b) \in A \times B}.$$

Using the shorthand reading, when the proof-objects are left out, we also see that the rule of \wedge -introduction is correct:

$$\frac{A \text{ true} \quad B \text{ true}}{A \wedge B \text{ true}}$$

For the \wedge -eliminations we need the use of the *projection-functions* p and q that are associated with the pairing-function. Consider the rule

$$\frac{A \wedge B \text{ true}}{A \text{ true}}$$

Restoring proof-objects, we see that from an element $c \in A \wedge B$, one has to find an element of A . But c is an element of $A \wedge B$, and so c is equal to (is a method for finding, can be evaluated, or executed, to) a *canonical* element $(a, b) \in A \wedge B$. Applying the projection p , we see that $p(c) = p((a, b)) = a \in A$, so the proper formulation will be

$$\frac{c \in A \wedge B}{p(c) \in A}$$

It should be mentioned, however, that the conjunction is not a primitive set-formation operation in the language of Martin-Löf. On the contrary, a suitable candidate can be defined from other sets and the appropriate rules derived.

A slightly more complex example is provided by the *universal quantification* $(\forall x \in A)B[x]$ and *implication* $A \rightarrow B$, both of which are treated as variants of the cartesian product $(\Pi x \in A)B[x]$ of a family of sets. This product may be formed only on the assumption that we have a family of sets over A , that is, provided that $B[x]$ is a set, whenever $x \in A$. Thus the formation rule will take the form

$$\frac{\begin{array}{c} x \in A^1 \\ \vdots \\ A \text{ set} \quad B[x] \text{ set} \end{array}}{(\Pi x \in A)B[x] \text{ set}}$$

(This serves to illustrate the important circumstance that the basic judgements may depend on assumptions. Better still, we should say that the right premise is a *hypothetical* judgement $B[x]$ set (provided that $x \in A$.) In order to understand the Π -formation rule one needs to know what a canonical element of $(\Pi x \in A)B[x]$ would be; this is told by the Π -introduction rule

$$\frac{\begin{array}{c} x \in A^1 \\ \vdots \\ b[x] \in B[x] \end{array}}{\lambda x. b[x] \in (\Pi x \in A)B[x]}$$

that is, the canonical elements are *functions* $\lambda x. b[x]$, such that $b[x] \in B[x]$

provided that $x \in A$. Just as in the case of conjunction, where the elimination rule was taken care of by matching the pairing function with a projection, one will obtain the elimination rule through a similar match between λ -abstraction and function-application, *ap*. Thus the rule takes the form

$$\frac{f \in (\Pi x \in A)B[x] \quad a \in A}{ap(F, a) \in B[a/x]}.$$

(In order to understand this rule one makes use of an important connection between abstraction and application, namely the law

$$ap(\lambda x.b[x], a) = b[a/x].$$

For the details of the explanation, refer to Martin-Löf [1982] or [1984]).

If the set (proposition) $B[x]$ does not depend on x the product is written as the set of functions B^A (as the proposition $A \rightarrow B$). The rules are obvious, with the exception of \rightarrow -formation:

$$\frac{\begin{array}{c} A \text{ true}^1 \\ \vdots \\ A \text{ prop} \end{array} \quad \begin{array}{c} B \text{ prop} \end{array}}{A \rightarrow B \text{ prop}}^1$$

Here the formation rule is stronger than the usual rule (where A and B both have to be propositions) because the right premise is weaker in that B has to be a proposition only when A is true. This concept of implication has been used by Stenlund in an elegant theory of definite descriptions, cf. his [1973] and [1975].

The other quantification is taken care of by means of the *disjoint union* of a family of sets. The Σ -formation rule takes the form

$$\frac{\begin{array}{c} x \in A^1 \\ \vdots \\ A \text{ set} \end{array} \quad \begin{array}{c} B[x] \text{ set} \end{array}}{(\Sigma x \in A)B[x] \text{ set}}^1$$

The canonical elements are given by the Σ -introduction rule

$$\frac{a \in A \quad b \in B[a/x]}{(a, b) \in (\Sigma x \in A)B[x]}$$

On the propositional reading, where the disjoint union is written as the

quantifier $(\exists x \in A)B[x]$, we see that in order to establish an existence claim one has to (i) exhibit a suitable witness $a \in A$ and (ii) supply a suitable proof-object b that the witness $a \in A$ does, in fact, satisfy the condition imposed by $B[x]$. The inclusion of the proof-object b allows yet a third use for the disjoint union, namely that of restricted comprehension-terms. What would, on a constructive reading, be meant by 'an element of the set of x 's in A such that $B[x]$ '? At least one would have to include a witness $a \in A$ and information (= a proof-object) establishing that a satisfies the condition $B[x]$. Thus the canonical elements of the restricted comprehension-term $\{x \in A : B[x]\}$ coincide with the canonical elements of the disjoint sum. This representation of 'such that' provides the key to the actual development of, say, the theory of real numbers given the set N of natural numbers. A real number will be an element of N^N such that it obeys a Cauchy-condition.

At this point I will refrain from further development of the language and instead I shall apply the type-theoretic abstractions that have been introduced so far to the notorious "donkey-sentence"

(*) Every man who owns a donkey beats it.

The problem here is, of course, that formulations within ordinary predicate logic do not seem to provide any way to capture the back-reference of the pronoun 'it'. A simple-minded formalization yields

(**) $\forall x (\text{Man}(x) \wedge \exists y (\text{Donkey}(y) \wedge \text{Own}(x, y)) \rightarrow \text{Beats}(x, ?))$.

There seems to be no way of filling the place indicated by '?', as the donkey has been quantified away by 'y'.

Using the disjoint-union manner of representation for restricted comprehension-terms one finds that 'a man who owns a donkey' is an element of the set

$\{x \in \text{MAN} : (\exists y \in \text{DONKEY})\text{OWN}[x, y]\}$.

Such an element, when in canonical form, is a pair (m, b) , where $m \in \text{MAN}$ and b is a proof-object for $(\exists y \in \text{DONKEY})\text{OWN}[m/x, y]$. Thus b , in its turn, when brought to canonical form, will be a pair (d, c) , where d is a DONKEY and c a proof-object for $\text{OWN}[m/x, d/y]$. Thus for an element z of the comprehension-term 'MAN who OWNS a DONKEY' the left projection $p(z)$ will be a man and the right projection $q(z)$ will be a pair whose left projection $p(q(z))$ will be the witnessing donkey. Putting it all together we get the formulation

(***) $(\forall z \in \{x \in \text{MAN} : (\exists y \in \text{DONKEY})\text{OWN}[x, y]\})$
 $\text{BEAT}[p(z), p(q(z))]$.

In this manner, then, the type-theoretic abstractions suffice to solve the problem of the pronominal back-reference in (*). It should be noted here that there is nothing *ad hoc* about the treatment, since all the notions used have been introduced for mathematical reasons in complete independence of the problem posed by (*). On the other hand one should stress that it is not at all clear that one can export the 'canonical proof-objects' conception of meaning outside the confined area of constructive mathematics. In particular, the treatment of atomic sentences such as 'OWN[x, y]' is left intolerably vague in the sketch above and it is an open problem how to remove that vagueness.

Martin-Löf's type theory has attracted a measure of metamathematical attention. Peter Aczel [1977, 1978, 1980, 1982], in particular, has been a tireless explorer of the possibilities offered by the type theory. Other papers of interest are Diller [1980], Diller and Troelstra [1984] and Beeson [1982].

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Note added in proof (October 1985)

Per Martin-Löf's 'On the meanings of the logical constants and the justifications of the logical laws' in *Atti degli incontri di logica matematica vol. 2*, Scuola di Specializzazione in Logica Matematica, Dipartimento di Matematica, Università di Siena, 1985, pp. 203–281, was not available during the writing of the present chapter. In these lectures, Martin-Löf deals with the topics covered in Sections 4–6 above in great detail and carries the philosophical analysis considerably further.

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