What is an Expression?*

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“I always have a quotation for everything — it saves original thinking.”
Lord Peter Wimsey, in: Dorothy L. Sayers, Have His Carcase, Chapter IV.

1. Consider the following explanation:

(1) $\bot$ is a proposition, namely the proposition that is true under no circumstances;

(2) $(A \rightarrow B)$ is a proposition, when $A$ and $B$ are, namely the proposition that is true on the condition that $B$ is true, provided that $A$ is true.

With this explanation I have accomplished an achievement comparable to that attempted by Frege in the Grundgesetze and defined a (formal) language $L$, the propositions of which can serve as contents of assertions and accordingly can be used for communication.¹

For contrast, consider also the following inductive definition, where the asterisk indicates the concatenation operation in the freely generated algebra of strings:

(i) $\bot$ is a WFF;

(ii) If $A$ and $B$ are WFF’s, then so is $(A \rightarrow B)$.

(iii) The only WFF’s are those obtained by (i) and (ii).²

With this definition I have accomplished an achievement comparable to that of Hilbert, and defined a metamathematical object-language $OL$, con-

¹ Comparable in that also my language is interpreted, but not as regards the scope of the endeavour. Frege wanted to give a formal language that was adequate for the practice of mathematical analysis. As is well-known, his attempt failed owing to the emergence of the Zermelo-Russell paradox. My present effort, on the other hand, is demonstrably successful, but of a very limited range.

² NB. The brackets and the horseshoe are here boldface.
cerning which metamathematical theorems can be proved. The explanation of the language $L$ has no extremal clause and it is in principle open-ended, just like the formal languages of Frege, Lesniewski and Martin-Löf; nothing rules out there being other propositions than those provided by the explanations (1) and (2). In the case of $OL$, on the other hand, the extremal clause (iii) serves to confine its WFF’s to those provided by (i) and (ii).

2. In spite of their superficial *formal* similarity, there is a considerable difference between the two examples. The expressions in the language $L$ can be used for stating truths. The WFF’s of $OL$ are not suitable tools for speaking or writing; they do not lend themselves for communication. Thus,

\[(\bot\vdash(\bot\vdash(\bot\bot))\vdash((\bot\bot)\bot))\]

is a meaningful declarative sentence that can be used for effecting acts of assertion. The WFF

\[(\ast\bot\ast(\ast\bot\ast(\ast\bot\ast\bot)\ast)\ast(\ast\bot\ast\bot)\ast(\ast\bot\ast)\ast(\ast\bot\ast)\ast)\]

on the other hand, is only an *object* to be talked about and does not lend itself for carrying out assertions or other communicative acts. This important circumstance is often ignored owing to the written nature of both items. WFF’s are objects, *(meta)mathematical objects*, and, as such, only meant to be spoken about. Metamathematical object-languages so called are not properly languages; they do not comprise expressions that can be used for communication. In early authoritative expositions of Gödel’s theory object languages are never exhibited. Since object-language “expressions” are never – indeed *cannot* be – employed to say anything, but are only spoken about, the object-language is *de facto* abolished. Instead, using clauses such as

A VARIABLE is a number of the form $29+4k$, $k=0,1,2,\ldots$

one proceeds directly to the Gödel numbering, bypassing the object-language entirely. The object-*language* “expressions” are treated *solely* as objects. Donald Monk’s excellent textbook is a telling example from within the metamathematical school of Tarski. A first-order language is nothing but a mathe-

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3 Even more comparable to the syntactic achievements of Gödel [1931], Tarski [1933], [1935], Hans Hermes [1938], Karl Schröter [1941] (“semitische Quadripen”).

4 For instance, those of Mostowski [1952, p. 27] and Feferman [1960].

5 In Wittgenstein’s Tractarian (3.328) words: “If a sign is *without use*, it has no meaning. This is the point of Ockham’s maxim”.

mathematical object, namely a certain quintuple construed set-theoretically. A natural, and possibly inevitable, culmination of this tendency can be found in Yuri Manin’s unorthodox but very stimulating Course in Mathematical Logic, where the expressive side of language is simply ignored:

Let $A$ be any abstract set. We call $A$ an alphabet. Finite strings of elements of $A$ are called expressions. Finite sequences of expressions are called texts.

Accordingly any finite string (over any set) is taken as an expression; hence, strictly and literally anything goes: meaning is now of no concern whatever in a “text”.

3. Consider the following definition of a function $\Phi$ by means of metamathematical recursion over the set $\text{wff}$’s.

\begin{align*}
(i) \quad & \Phi(0) = \text{def} \perp \in \text{wff}; \\
(ii) \quad & \Phi(k+1) = \text{def} (\ast \Phi(k) \ast \ast \ast \Phi(k)) \in \text{wff}.
\end{align*}

Hence

\[ \Phi \in \mathbb{N}\rightarrow \text{wff}, \text{ and } \Phi(k+1) \in \text{wff}, \text{ for every } k \in \mathbb{N}. \]

However,

when $a \in \mathbb{N}$, then also $10^a \in \mathbb{N}$.

Accordingly, since $23 \in \mathbb{N}$,

\[ m = \text{def} (10^{10^{10^{10^{23}}}} + 1) \in \mathbb{N}, \]

where the exponentiation has been iterated 23 times, and so

\[ \Phi(m) \in \text{wff}. \]

This wff is a certain (meta)mathematical object. If we want to refer to it, however, there is no other way available to us than to use some such device as the $\Phi$-function. The definition of the set $\text{wff}$ runs completely parallel to the recursive definition of the natural numbers:

\[ 0 \in \mathbb{N} \]

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6 Monk [1976, Definition 10.1, p. 162]. Indeed, Monk’s “languages” are the counterparts of Schröter’s semiotische Quadrupel.

7 Manin [1977, p. 3].
and

\[
\frac{a \in \mathbb{N}}{s(a) \in \mathbb{N}}
\]

where the corresponding extremal clause is the principle of definition by recursion/proof by induction, with respect to the natural numbers thus defined. In Constructive Type Theory, a set is defined by laying down how its canonical elements may be formed, and when two canonical elements are equal canonical elements of the set.\(^8\) Non-canonical elements are equal when they evaluate to equal canonical elements. The canonical natural numbers are 0 and successors. The relevant specifications of equality are

\[
0 = 0 \in \mathbb{N},
\]

that is, 0 is the same canonical element of \(\mathbb{N}\) as 0, and

\[
\frac{a = b \in \mathbb{N}}{s(a) = s(b) \in \mathbb{N}}
\]

that is, when \(a\) and \(b\) are (not necessarily canonical) equal elements of \(\mathbb{N}\), their successors are equal canonical elements of \(\mathbb{N}\). There is no way that we can give the non-canonical element \(\Phi(m)\) of the set \(\text{wff}\) in a fully canonical form, without non-canonical parts. The string \(\Phi(m)\) over the alphabet \{\(\bot, \lor, (, )\}\) has a length that is larger than (it is customary to claim on these occasions) the number of atoms in the known universe. On the other hand, \(\Phi(m)\) is a real expression that stands for a metamathematical “expression”, namely the (non-canonical) element \(\Phi(m)\) of the set \(\text{wff}\). Metamathematical expressions do not express, owing to their lack of content; they are objects that can be expressed, using real expressions with meaning, for instance, as just mentioned, the expression \(\Phi(m)\).

If I want to use an expression (that \textit{eo ipso} has meaning, owing to its nature), I cannot get by with a simulacrum such as a name of the expression; I need the expression itself and not its name. The name of a metamathematical \(\text{OL}\) “expression”, on the other hand, is a real expression with content, but what it names is an object without meaning, namely an element of a freely generated algebra of strings.

Some might find it unnecessary or otherwise unprofitable to harp on this difference; what does it matter?\(^9\) One then does well to recall that Tarski’s

\(^8\) Martin-Löf [1984, pp. 8-10]

\(^9\) I know personally quite a few philosophers who do; strangely enough, a surprising number of them have names (first or otherwise) that begin with the letter D.
[1933] truth-predicate is a propositional function with the set of WFF's as its range of significance:

$$T(x)$$ is a proposition, provided $$x \in \text{wff}.$$\(^{10}\)

But

$$\left( \bot \rightarrow \bot \right) \in \text{wff},$$

and thus

$$T\left( \left( \bot \rightarrow \bot \right) \right)$$ is a proposition.

Furthermore, it is a true proposition, since

$$[T(\left( \bot \rightarrow \bot \right)) \leftrightarrow (\bot \supset \bot)]$$ is true,

since $$(\bot \supset \bot)$$ is a true proposition, both constructively, with proof-object $$\lambda x. x$$, that is, the identity function on the (empty) set of proofs of $$\bot$$, and non-constructively in virtue of the truth-table for $$\supset$$.

An attempt to apply the Tarskian T-predicate would, however, turn into a fiasco. As already stressed, the range of significance for the propositional function $$T(x)$$ consists of WFF's rather than propositions; an attempt to insert the proposition $$T(\left( \bot \rightarrow \bot \right))$$ into the argument place held by the variable that ranges over WFF's would rift asunder the categorical structure of our language and, concomitantly, also that of the world.

4. Thus far, I have only trodden the via negativa when attempting an answer to the question of my title. An essentially correct – albeit trivial – positive answer is readily forthcoming though: whatever an expression may be, it must express, or have, content. The expression is an expression only in virtue of expressing what it expresses. An expression without meaning – an expression that does not express – is a contradiction in terms. With respect to real (“meaningful”) expressions we accordingly have the following formulae:

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10 Strictly speaking this is not true, since Tarski does not employ a typed universe of individuals. Accordingly, his T-predicate, like any other predicate, has the universe of everything as its range of significance. Propositions, nevertheless, also for Tarski, are not things. Thus, the point I make below holds good also for his formulation; that his object language – for the “general theory of classes” – is considerably richer than mine is of no concern here.
expresses
\[ 'c' \] means \[ c \]
has the meaning

that is,

\[ c \] is expressed by \[ 'c'. \]

It is important to realise that the expression \('c'\) is obtained from the meaning \(c\). Without the meaning \(c\) being given in this way, there is no access to the expression \('c'\). The expression is obtained from the meaning by disregarding what is given, namely the meaning (object) \(c\) and instead focusing on how (in what form) it is given, namely through the expression \('c'\). If we consider the object (meaning) \(c\) and divest it of content what remains is the form \('c'\) through which it is given:

formalization \( \downarrow \)
\[ 'c' \]
“contentualization”,
sugaring

Only in virtue of its meaning-bearing role do we have access to the expression: when nothing is expressed, there is no expression. Wittgenstein in the Tractatus would say that the relation between the expression ("symbol") and what it expresses, or symbolises, is internal. For instance, somewhat later, concerning propositions, that is a special kind of expression, we find:

Ogden & Richards and Russell consider that the relation of proposition to fact is an external relation; this is not correct. It is an internal relation. An internal relation cannot be otherwise; it is given in the terms involved, in the nature of proposition and fact.\(^\text{11}\)

The modern ("Quinean") relation of reference

\[ c \text{ refers to } b \text{, in symbols Ref}(c, b), \]

on the other hand, is an external one that may, or may not, obtain.\(^\text{12}\) The

\[^{11}\text{Wittgenstein [1930-32, Lecture A IV, p. 9].}\]

\[^{12}\text{Some writers in the modern "semantic" tradition here prefer Val}(c) = b, \]

using a "valuation" function \(\text{Val}\) instead of the reference relation \(\text{Ref}.\) Mutatis mutandis the points made remain valid.
objects c and b are independent of each other and can exist without standing in the Ref relation. For Quine, reference is

an important relation of words to objects – or better – of words to other objects, some of which are not words – or even better, of objects some of which are words to objects some of which are not words.\footnote{Nelson Goodman's apt words in Quine [1973, p. xi].}

Without the object/meaning c, the expression, or “word”, ‘c’, that is, the material embodiment of the meaning c, cannot exist, however; in the other direction, as soon as the meaning c is given, or “expressed”, we have embodied the meaning c in the expression ‘c’.

Frege, the Founding Father of modern logic, for one, was fully aware of these issues and held the right views:

As Sinn causes difficulties at times, one quickly decides to throw it out and then naturally retains the soulless signs. The progenitor of such a theory does not want to express Thoughts with his signs, but only to play according to certain rules. Thus, in no way truth is there at issue. ... One empties the signs of content, in order to avoid inconvenient questions; nevertheless, one then also balks at acknowledging the signs as really empty.\footnote{Frege [1906, II, p. 396]: Da der Sinn zuweilen Schwierigkeiten macht, treibt man ihn kurz entschlossen ganz aus und behält dann natürlich die entseelten Zeichen zurück. Der Urheber einer solchen Theorie will mit seinen Zeichen keine Gedanken ausdrücken, sondern nur nach gewissen Regeln spielen. Also kann es dabei gar nicht um Wahrheit handeln. ... Man entleert die Zeichen, um unbequemen Fragen zu entgehen; aber die Zeichen dann auch wirklich als leer anzuerkennen, sträubt man sich.}

From earlier formulations in Über Sinn und Bedeutung we know that Frege uses bezeichnen (designate) and bedeuten (signify) for the relation between the Zeichen (sign) and its Bedeutung.

Ein Eigenname (Wort, Zeichen, Zeichenverbindung, Ausdruck) drückt aus seinen Sinn, bedeutet oder bezeichnet seine Bedeutung. Wir drücken mit einem Zeichen dessen Sinn aus und bezeichnen mit ihm dessen Bedeutung.\footnote{[1892, p. 34]}

In view of the ambiguity in Bedeutung, perhaps first noted by Ignacio Angelelli, signification (significance?) might well be the best translation into...
English, covering both of Angelelli’s readings: designated entity and logical significance (weight, importance). Frege’s formula then becomes:

a sign expresses its sense and designates, or signifies, its signification.

The sense, furthermore, should, according to Frege, “contain a mode of givenness” for the signification:

Es liegt nun Nahe, mit einem Zeichen (Namen, Wortverbindung, Schriftzeichen) außer dem Bezeichneten, was die Bedeutung des Zeichen heißene möge, noch das verbunden zu denken, was ich den Sinn des Zeichens nennen möchte, worin die Art des Gegebenseins enthalten ist.

In these formulations, the sign cannot be of the soulless kind, since then nothing is expressed. It is the sign with content (Inhalt) that expresses a sense (Sinn) containing (enthalten) a mode of givenness of, or for, the designated entity.

Ferdinand de Saussure must have developed his theory of the sign during the first decade of the twentieth century. His signe comprises two parts, the material signifiant and the conceptual signifié:

I call the combination of a concept and a sound-image a sign, but in current usage the term generally designates only a sound-image, a word, for example (arbor, etc.). One tends to forget that arbor is called a sign only because it carries the concept “tree”, with the result that the idea of the sensory part implies the idea of the whole.

... Ambiguity would disappear if the three notions involved here were designated by three names, each suggesting and opposing the others. I propose to retain the word sign [signe] to designate the whole and to replace concept and sound-image respectively by signified [signifié] and signifier [signifiant]; the last two items have the advantage of indicating the opposition that separates them from each other and from the whole of which they are parts. As regards sign, if I am satisfied with it, this is simply because I do not know of any word to replace it, the ordinary language suggesting no other.

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16 Angelelli [1982] and earlier writings cited there.
17 [1892, p. 31].
18 [1892, p. 26].
19 De Saussure [1916, p. 67].
Both Frege:

\[
\begin{array}{c}
\text{Sinn} \\
\text{Zeichen} \rightarrow \text{Ding (Bedeutung)}
\end{array}
\]

and De Saussure:

\[
\begin{array}{c}
\text{Concept} \\
\text{Acoustic image} \rightarrow \text{Thing}
\end{array}
\]

get semiotic triangles that are strongly reminiscent of the Scholastic tag *vox significat mediantibus conceptibus*, with the corresponding semiotic triangle:

\[
\begin{array}{c}
\text{conceptus} \\
\text{vox} \rightarrow \text{res}
\end{array}
\]

The similarity is obvious. The diagrams are isomorphic: Frege’s *soulless sign* corresponds to De Saussure’s *acoustic image*, which is the Scholastic *vox*, and the (*mediating*) *concept* employed by De Saussure (and the Scholastics) corresponds to Frege’s *Sinn*.\(^\text{20}\)

For De Saussure, the sign is comprised out of two *interlocking* parts, namely the acoustic image and the concept. Accordingly, if we allow ourselves to be guided by the above correspondence, the Fregean sign would comprise the soulless sign and its sense. How, precisely, the sign expresses its sense, which, in its turn, contains a route to the designated entity, I do not see clearly. On the other hand, Frege’s brief elucidation, in the by now classical essays from the 1890’s, might not suffice for determining the precise relationship between the soulless sign, the sign, its sense, and designated entity, nor, indeed, were they

\(^\text{20}\) The anti-psychologist Frege [1892, p. 29] would, however, reject De Saussure’s mentalistic notion of a concept; his sense is a Platonist “third realm” denizen.
necessarily intended by Frege to do so.\textsuperscript{21} Such underdetermination would not be surprising, given the modest stated task of those remarks, namely that of buttressing the early sections of the \textit{Grundgesetze der Arithmetik}. On his own perception, Frege was offering aid towards understanding the technical work. Nothing indicates that he considered himself to have (created) a thoroughly worked out philosophy of language, including a complete semiotic.

Concerning the connection between sign and sense, Wittgenstein, in the \textit{Tractatus}, is reasonably explicit, and certainly more so than Frege. A very late change, and possibly the last substantial one, in the step from the \textit{Prototractatus} to the \textit{Tractatus}, introduces the distinction between sign and symbol.\textsuperscript{22} A symbol comprises a material part, the sign, in which the symbol is \textit{sinnlich wahrnehmbar} (3.32). This part is \textit{accidental} (3.34). Initially, any other choice would have done, but once the choice is made one has to live with it. Another aspect of the symbol is \textit{essential}, namely what is common to all symbols that can serve the same purpose (3.341). For instance, the English colour-word \textit{red} can serve the same purpose as the German \textit{rot}. For Wittgenstein, \textit{"(P&Q)"} and \textit{"(~P\lor~Q)"} are the same symbol, because they serve the same purpose, namely that of effecting a certain (directed) partition in logical space, in virtue of having the same truth-table. That truth-table, incidentally, is yet a third way, according to Wittgenstein, of giving the same symbol. In all three cases, the (perceptible) sign component of the symbol differs from those of the other two, but the symbol, nonetheless, is the same.

Accordingly the Tractarian semiotic triangle is a more complex one:

\begin{center}
\textit{Symbol:}
\end{center}

\begin{center}
\begin{tikzpicture}
\node (c) at (0,0) {Zeichen};
\node (h) at (0,-1) {\textit{das Wesentliche am Symbol}};
\node (l) at (0,-2) {\textit{das Projizierte}};
\draw[->] (c) -- (h);
\draw[->] (c) -- (l);
\end{tikzpicture}
\end{center}

In order to spell out the connections between the sign (\textit{Zeichen}), symbol and what is essential in the symbol (\textit{das Wesentliche am Symbol}), that is, the

\footnote{The footnotes [1893, pp. 5, 7, and 9] as well as the \textit{Vorwort} to [1891] are relevant here.}

\footnote{Kremer [1997, pp. 101-107] is very helpful on these matters; in particular, he notes the connection to Sellars, more about which in the sequel.}
linguistic role fulfilled by the sign, we may profitably employ notions and notations introduced by Wilfrid Sellars. Ordinary quotes are used for symbols; thus, ‘red’ is an English expression with meaning, that is, a symbol. Asterisk quotes are used for material aspects of symbols, e.g. *red* is a sign. Finally, dot quotes are used to indicate linguistic roles. Accordingly, the linguistic role •red• is played in Swedish by some *röd*’s. For all three notions a type/token distinction applies: for instance, the token

red

is a *red*, and so is also the token

red.

Employing Sellars’ notation to a more or less concrete example, our diagram takes the form:

\[\text{‘house’:\begin{tikzpicture}
  \node[circle, draw] (house) {•house•};
  \node[below] at (house) {*house*};
  \draw[->] (house) -- (house); \end{tikzpicture}}\]

An entirely parallel semiotic pattern is obtained also in De Saussure’s *Cours de linguistique générale*: according to the correspondence:

\[
\text{De Saussure: signe — signifiant — signifié;}
\]

\[
\text{Wittgenstein: Symbol — Zeichen — das Wesentliche am Symbol.}
\]

We may also attempt to capture this semiotic structure in a

**Neutral Terminology:** sign — form — content.

Our semiotic structure can be applied also to non-linguistic matters. The game of chess serves well to illustrate this, as De Saussure probably was the

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23 See Sellars [1950], [1963] and above all [1974, §§ IV-VI].
first to stress.24 This chess example is by now very well-known, owing to its being used in Wittgenstein’s *Philosophische Untersuchungen* (§ 108). The white King — the chess piece, that is — in one sense is a strangely shaped piece of varnished wood. This piece of wood, which corresponds to De Saussure’s acoustic image and to Wittgenstein’s accidental aspect of the symbol, may of course serve in many roles, and can be used for many purposes, among which those given by:

1. “Blunt instrument to be entered into evidence as Exhibit B, m’lud.”
2. (unsuccessful) specimen of peasant religious art;
3. barbecue ignition-fuel;
4. a toy church-tower from a children’s’ box of bricks;
5. agglomeration of vanished carbon compounds (a chemical specimen);
6. totem in a phallic religion;
7. the Chess King.25

Why content has been largely lost in modern philosophy of language, why signs are held to be merely (the analogues of) dead wood, I cannot fathom. The conflation between real expressions (with content, and used for speaking) and metamathematical “expressions” (without content, only to be spoken about) that is often found in logic since the metamathematical turn around 1930 is, of course, a possible reason for, as well as a symptom of, the lack of content. Another is Quine’s preference for objects in conjunction with his hostility to meaning that has led him to view expressions, say, the bearers of reference, solely as *objects*. His preferred relata in the reference relation are material objects. Again, content is lost, but for Quine, this is neither unexpected nor unwelcome.

I have been concerned to spell out a difference between words (“expressions”) and objects: words just are not just objects. Respecting this difference has far-reaching consequences for how semantics should be done. For instance, it is not at all clear that current theories of reference that are patterned on the evaluation-functions for terms in Tarskian metamathematical theories of truth are able to serve as Fregean theories of reference. And how do the various approaches to definite descriptions fare when one takes the difference between word and object seriously? An elaboration of a certain *Proper Names Theory for Definite Descriptions* (!!!) that flows naturally on my approach will however have to wait for another occasion, so as not to tax the Editorial patience beyond endurance.

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24 The references to chess are listed in the index in the English (1967) translation; Part II, Chapter 3 is particularly relevant here. Harris [1988, pp. 24-26] notes the parallel between De Saussure and Wittgenstein.

25 According to some, among whom Chess World Championship Candidate (joint winner of the AVRO Tournament 1938) and psychoanalyst Reuben Fine [1967, pp. 8-12], the last two items (6) and (7) coincide.
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