Universality of weak localization in disordered wires

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We compute the quantum correction $\delta A$ due to weak localization for transport properties $A = \sum_n a(T_n)$ of disordered quasi-one-dimensional conductors, by integrating the Dorokhov-Mello-Pereyra-Kumar equation for the distribution of the transmission eigenvalues $T_n$. The result $\delta A = \left(1 - 2/\beta\right) a(1) + \int_0^\infty dx \frac{a(\cosh^{-2} x)}{4x^2 + \pi^2}$ is independent of sample length or mean free path, and has a universal $1 - 2/\beta$ dependence on the symmetry index $\beta \in \{1, 2, 4\}$ of the ensemble of scattering matrices. This result generalizes the theory of weak localization to all linear statistics on the transmission eigenvalues.

Weak localization is a quantum transport effect which manifests itself as a magnetic-field-dependent correction to the classical Drude conductance. Discovered in 1979, it was the first known quantum interference effect on a transport property. (For reviews, see Ref. 6.)

The starting point of the analysis is the Dorokhov-Mello-Pereyra-Kumar equation

$$\frac{\partial P}{\partial s} = \frac{2}{\beta N + 2 - \beta} \sum_{i=1}^N \frac{\partial}{\partial \lambda_i} \lambda_i (1 + \lambda_i) J \frac{\partial}{\partial \lambda_i} J^{-1} P$$

for the evolution of an ensemble of quasi-1D conductors of increasing length. For each ratio $s = L/l$ the ensemble is characterized by the probability distribution $P(\{\lambda_n\}, s)$ of the set of variables $\{\lambda_n\} = \lambda_1, \lambda_2, \ldots, \lambda_N$. The $\lambda$-variables are defined by $\lambda_n = (1 - T_n)/T_n$ in terms of the transmission eigenvalues $T_n$. Since $T_n \in [0, 1]$, $\lambda_n \in [0, 1]$.

The purpose of this paper is to demonstrate that the universality of the weak-localization correction expressed by Eq. (1) is generic for a whole class of transport properties, of which the conductance is but a special example. We consider a general transport property $A$ of the form

$$A = \sum_{n=1}^N a(T_n).$$

This is the definition of a linear statistic on the transmission eigenvalues $T_1, T_2, \ldots, T_N$. The word "linear" indicates that $A$ does not contain products of different $T_n$'s, but the function $a(T)$ may well depend nonlinearly on $T$. The conductance is a special case for which $a(T) = (2e^2/h)T$ is linear in $T$ (Landauer's formula). Other examples of linear statistics include the shot-noise power [with $a(T)$ a quadratic function], the conductance of a normal-superconductor interface [with $a(T)$ a rational function], and the supercurrent through a point-contact Josephson junction [with $a(T)$ an algebraic function]. In Ref. 6 it was shown that the theory of "universal conductance fluctuations" can be generalized to all these linear statistics. Here we wish to establish such generality for the theory of weak localization.

Our final result is a formula

$$\delta A = \left(1 - 2/\beta\right) a(1) + \int_0^\infty dx \frac{a(\cosh^{-2} x)}{4x^2 + \pi^2}$$

for the weak-localization correction $\delta A$ to the ensemble average $\langle A \rangle = A_0 + \delta A$ of an arbitrary linear statistic $A$ of the form (2). The term $\delta A$ is a quantum correction of order $N^\beta$ to the classical $\beta$-independent value $A_0$, which is of order $N$ (with $N \gg 1$ being the number of scattering channels in the conductor). One easily verifies that substitution of $a(T) = (2e^2/h)T$ into Eq. (3) yields the known result (1), using

$$\int_0^\infty dx \frac{a(\cosh^{-2} x)}{4x^2 + \pi^2} = \frac{1}{12}.$$
\[ P = P_0 + \delta p \text{ of order } 7V^0, \]

with \( P_0 \) of order \( 7V \) and \( \delta p \), neglecting terms of order \( TV^0 \). To this end we decompose

\[ \text{Substitution into Eq. (9) yields to order } 7V \text{ an equation} \]

\[ x' = -\ln |\sinh^2 x|, \]

\[ \text{with the definitions } \gamma = 1 - 2/\beta, V(x) = -\ln [\sinh 2x], \]

\[ u(x, x') = -\ln [\sinh^2 x - \sinh^2 x']. \]

We need to solve Eq. (9) to the same order in \( N \) as the expansion (7), i.e., neglecting terms of order \( N^{-1} \). To this end we decompose \( \tilde{\rho} = \rho_0 + \delta \tilde{\rho} \), with \( \rho_0 \) of order \( N \) and \( \delta \tilde{\rho} \) of order \( N^0 \). Substitution into Eq. (9) yields to order \( N \) an equation

\[ \frac{\partial \tilde{\rho}_0}{\partial s} = \frac{1}{2N} \frac{\partial}{\partial x} \tilde{\rho}_0 \frac{\partial}{\partial x} \int_0^\infty dx' \tilde{\rho}_0(x', s) u(x, x'). \]

This is essentially the problem solved by Mello and Pichard,\textsuperscript{13} who showed that

\[ \tilde{\rho}_0(x, s) = N s^{-1} \theta(s - x), \]

in the relevant regime \( s \gg 1, s \gg x \). [The function \( \theta(\xi) \) equals 1 for \( \xi > 0 \) and 0 for \( \xi < 0 \).] Equation (11) implies that, to order \( N \), the \( x \)-variables have a uniform density of \( N/L \), with a cutoff at \( L/l \) such that \( \int_0^\infty dx \tilde{\rho}_0 = N \). In the cutoff region \( x \sim L/l \) the density deviates from uniformity, but this region is irrelevant since the transmission eigenvalues are exponentially small for \( x \gg 1 \). One can readily verify by substitution that the solution (11) satisfies Eq. (10), using

\[ \frac{\partial \delta \tilde{\rho}}{\partial s} \int_0^s dx' u(x, x') = -2x \text{ for } s \gg 1, s \gg x. \]

Now we are ready to compute the \( O(N^0) \) correction \( \delta \tilde{\rho} \) to the density. Substituting \( \tilde{\rho} = \rho_0 + \delta \tilde{\rho} \) into Eq. (9), and using Eqs. (11) and (12), we find

\[ \frac{\partial \delta \tilde{\rho}}{\partial s} = \frac{1}{2s} \frac{\partial^2}{\partial x^2} \int_0^\infty dx' \delta \tilde{\rho}(x', s) u(x, x') \]

\[ -\frac{1}{s} \frac{\partial}{\partial x} (x \delta \tilde{\rho}) - \frac{\gamma}{4s} \frac{\partial^2 V}{\partial x^2} - \frac{\gamma}{s}. \]

The last term \( \gamma/s^2 \) on the right-hand side is a factor \( s \) smaller than the other terms, and may be neglected for \( s \gg 1 \). Equation (13) thus has the \( s \)-independent solution \( \delta \tilde{\rho}(x) \) satisfying

\[ \frac{1}{2} \frac{d^2}{dx^2} \int_0^\infty dx' \delta \tilde{\rho}(x') \ln |\sinh^2 x - \sinh^2 x'| \]

\[ + \frac{d}{dx} \left[ x \delta \tilde{\rho}(x) \right] = \frac{\gamma}{4} \frac{d^2}{dx^2} \ln |\sinh 2x|. \]

It remains to solve the integro-differential equation (14). This can be done analytically by means of the identity\textsuperscript{14}

\[ \int_0^\infty dx' f(x') \ln |\sinh^2 x - \sinh^2 x'| = \int_{-\infty}^\infty dx' f(|x'|) \ln |\sinh(x - x')|, \]

which transforms the integration into a convolution. The Fourier transform then satisfies an ordinary differential equation, which is easily solved. The result is

\[ \delta \tilde{\rho}(x) = (1 - 2/\beta)[\frac{1}{2} \delta(x) + (4x^2 + \pi^2)^{-1}], \]

as one can also verify directly by substitution into Eq. (14). The correction (16) to the uniform density (11) takes the form of a deficit (for \( \beta = 1 \)) or an excess (for \( \beta = 4 \)), concentrated in the region \( x \lesssim 1 \). For \( \beta = 2 \)
there is no $O(N^0)$ deviation from uniformity. The existence of a $\beta$-dependent density excess or deficit in the metallic regime was anticipated by Stone, Mello, Mutalib, and Pichard $^{10}$ from the $\beta$-dependence of the localization length in the insulating regime. However, as emphasized by these authors, their argument is simply suggestive and needs to be made quantitative. Equation (16) does that.

The weak-localization correction $\delta A$ follows upon integration,

$$\delta A = \int_0^\infty dx \delta \rho(x) a(1/\cosh^2 x). \quad (17)$$

Combination of Eqs. (16) and (17) finally gives the formula (3) for the weak-localization correction to the ensemble average of an arbitrary linear statistic, as advertised in the introduction.

We conclude with an illustrative application of Eq. (3). to the conductance $G_{\text{NS}}$ of a disordered normal-metal-superconductor (NS) junction. This transport property is a linear statistic for zero magnetic field. In a semiclassical treatment, the ensemble average $\langle G_{\text{NS}} \rangle$ is just the Drude conductance — unaffected by Andreev reflection at the NS interface. However, the quantum correction $\delta G_{\text{NS}}$ due to weak localization is enhanced by Andreev reflection. $^{15}$ Previously, there was no method to calculate $\delta G_{\text{NS}}$. $^8$ Now, using Eq. (3) one computes (for $\beta = 1$) $\delta G_{\text{NS}} = (1 - 4\pi^{-2})(2e^2/h)$, which exceeds the result $\delta G = -\frac{1}{3}(2e^2/h)$ in the normal state by almost a factor of 2. The experimental observation of the enhancement of weak localization by Andreev reflection has recently been reported. $^{16}

In summary, we have shown that the universality of the weak-localization effect in disordered wires is generic for a whole class of transport properties, viz., the class of linear statistics on the transmission eigenvalues. A formula has been derived which permits the computation of the weak-localization correction in cases that previous methods were not effective. This quantum correction is independent of sample length or mean free path, and has a $1 - 2/\beta$ dependence on the symmetry index, for all linear statistics.

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\begin{thebibliography}{16}

5. The wire geometry $L \gg W$ is essential for the universality of Eq. (1): in a square or cube geometry $\delta G$ acquires a dependence on $L$ and $l$, cf. Ref. 3. The zero-temperature limit is also essential: $\delta G$ becomes dependent on the phase-coherence length $l_\phi$ if $l_\phi < L$.
7. Although Ref. 6 also deals with a general theory of quantum transport for linear statistics, both the starting point and the goal are different. Reference 6 addresses the fluctuations around the ensemble average, by means of a random-matrix theory which by construction contains no information on the ensemble average. The present paper, in contrast, addresses the ensemble average itself, which is where the weak-localization effect plays a role.
8. Previous theoretical work on weak localization has either been based on diagrammatic perturbation theory, (Refs. 1–3), or on a moment expansion of Eq. (4) (Ref. 4). Both methods are unsuitable for arbitrary $a(T)$, and thus cannot be used for the present purpose. In fact, apart from the conductance, the only other quantity for which the weak-localization correction has been calculated previously is the shot-noise power; see M. J. M. de Jong and C. W. J. Beenakker, Phys. Rev. B 46, 13400 (1992).
12. F. J. Dyson, J. Math. Phys. 13, 90 (1972). The asymptotic expansion (7) was derived by Dyson for a problem where $\lambda$ is free to vary from $-\infty$ to $\infty$. In Ref. 6 it is shown that the restriction $\lambda \geq 0$ introduces no extra terms, to the order considered.
14. I am indebted to B. Rejaei for teaching me this method of solution.
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